Strictly according to the Syllabus of the
J. & K. University

NEW
UNIVERSITY TRIGONOMETRY

For

(Pre-University & Higher Secondary Classes)

~~I love you.~~

by

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Ali Mohammad & Sons,

BOOK SELLERS AND PUBLISHERS,
Habba Kadal, Srinagar, KASHMIR.

Published by :
Ali Mohammad & Sons,
Srinagar, Kashmir.

Price : Rs. 4.75

Printed at Rising Sun Press, Chawri Bazar, Delhi.

PREFACE

This book on Trigonometry has been written to meet the needs of the Pre-University students studying within the jurisdiction of the J. & K. University. Thus it covers the entire syllabus prescribed by the University. What the student has already studied in school or is expected to have studied there has not been touched at all. Moreover, unnecessary details likely to confuse the average student have been avoided as far as possible. Various articles have been explained in such a way that even the weakest student can grasp them provided that he studies these with care. Most of the articles have been illustrated by means of a number of solved examples most of which have been taken from University papers. In short, no pains have been spared to make the book intelligible and, at the same time, interesting.

The authors shall most thankfully receive any valuable suggestions or corrections that might have escaped their notice.

Srinagar
May, 1964.

Authors

Syllabus For The Higher Secondary Examination

Sexagesimal and circular units of angular measurements, Trigonometrical ratios and the simple relations connecting them; relations between Trigonometrical ratios of angles differing by multiples of right angles, additions and subtraction Formulae; Trigonometrical Ratios of multiple and sub-multiple angles. General solution of simple Trigonometrical equations: the relations between the sides and the angles of a triangle: logarithms, solution of triangles and simple cases of heights and distances, radii of the circumscribed, inscribed and escribed circles of triangles; areas of a triangle, regular polygon and of a circle; graphs of simple Trigonometrical Functions.

Syllabus For The Pre-University Examination

Relations between Trigonometrical ratios of angles differing by multiples of right angles, addition and subtraction formulae: Trigonometrical ratios of multiples and submultiples of angles. General solutions of simple Trigonometrical equations; the relations between the sides and the angles of a triangle, simple cases of heights and distances, radii of the circumscribed, inscribed and escribed circles of triangles, area of a triangle, regular polygon and the circle, graphs of simple Trigonometrical unctions.

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CHAPTER I

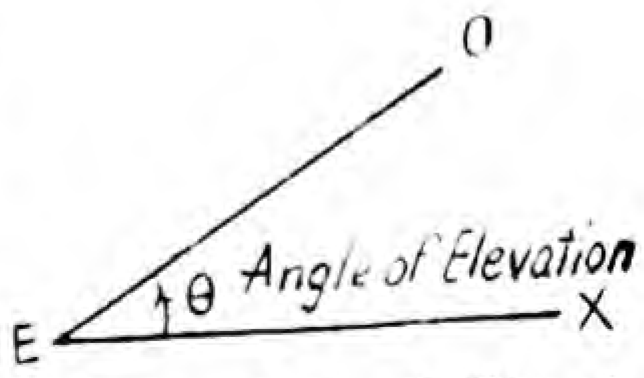
Heights And Distances

1.1 The student has already learnt a lot with regard to the definitions of *Trigonometrical Ratios.* He has also learnt some fundamental relations thereof, such as $\text{Sin}^2\theta+\text{Cos}^2\theta=1$, $1+\tan^2\theta=\text{Sec}^2\theta$, *etc.* We now propose to discuss in the present chapter one of the most interesting uses of Trigonometry, *viz.*, the finding of heights without actually measuring them, and finding of distances between two points without actually travelling. Thus it will be found that Trigonometry is very useful in measuring the heights and the distances of points which are otherwise inaccessible, for example, of the moon, the sun and the planets. For the solution of such problems, however, knowledge of some angles and distances is essential. The angles of objects are measured by an instrument known as *Theodolite.*

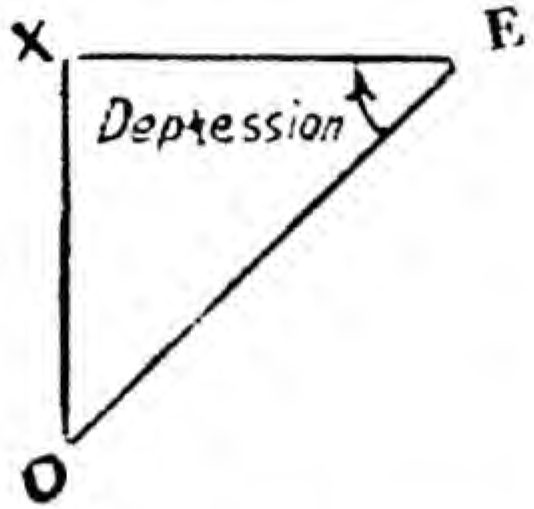
1.2 Before we illustrate the method of finding heights and distances, we define below *Angles of Elevation and Depression.*

Definition :—

Angle of Elevation. If O be an object at a *higher* level than E, the point of observation, then the angle $\angle XEO$ which EO, the st. line from the point of observation to the object observed makes with the horizontal line EX in the vertical plane OEX is called the *Angle of Elevation* of O as seen from E.



Angle of Depression.



If O be an object at a level lower than E, the point of observation, and EX be the horizontal line through E, then the angle $\angle$XEO, which EO, the st. line from the point of observation to the point observed, makes with EX is called the *Angle of Depression*, of O as seen from E.

Note :—The *Angle of Elevation* is sometimes called the *Altitude* of the object as well.

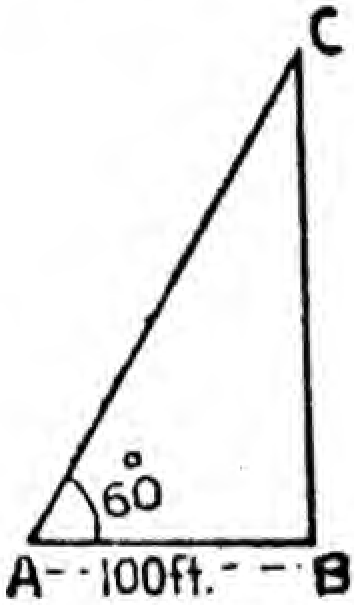
1.3. In working out problems on *Heights and distances*, we have to make frequent use of the Trigonometrical Ratios of some acute angles like 0°, 30°, 45°, *etc*., and the student is already expected to know their values. All the same, he is advised to go through the chart on page 3, which can give him all such information.

$\theta =$	0°	30°	45°	60°	90°
$\sin\theta =$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta =$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta =$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞
$\cot\theta =$	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0
$\sec\theta =$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞
$\operatorname{cosec}\theta$	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1

Note $\sqrt{2} = 1{\cdot}414$ *and* $\sqrt{3} = 1{\cdot}732$

Solved Examples

Ex. 1 A man standing 100 ft. away from the foot of a tower finds that the angle of elevation of the top is 60°. Find the height of the tower.



Sol. Let BC be the tower and A the observer.

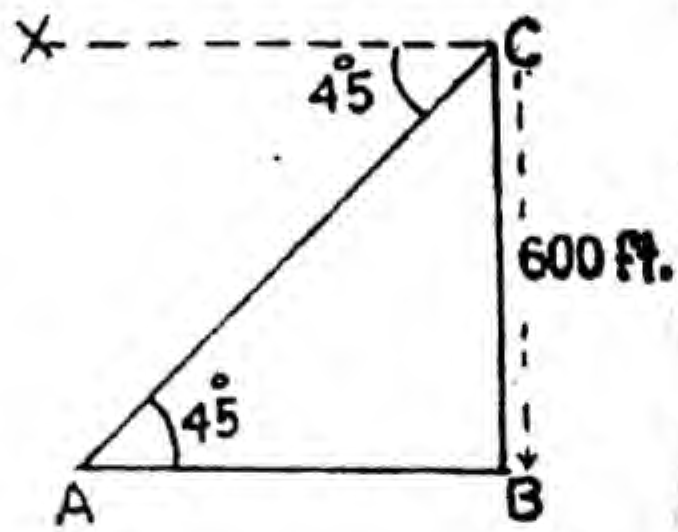
Then AB=100 ft. $\angle$BAC=60° and BC=h ft.=?

Now $\frac{BC}{AB}$=tan BAC or $\frac{h}{100}$=tan 60°=$\sqrt{3}$

or h=$100\sqrt{3}$ft.

Ex. 2. A cliff is 600 ft. high. A man observes a boat in a lake making angle of depression equal to 45°. Find the distance between the boat and the foot of the cliff.

Sol. Let C be the top of the cliff and A the boat.



Then $\hat{XCA}=45°$; BC=600 ft. and AB=x=?.

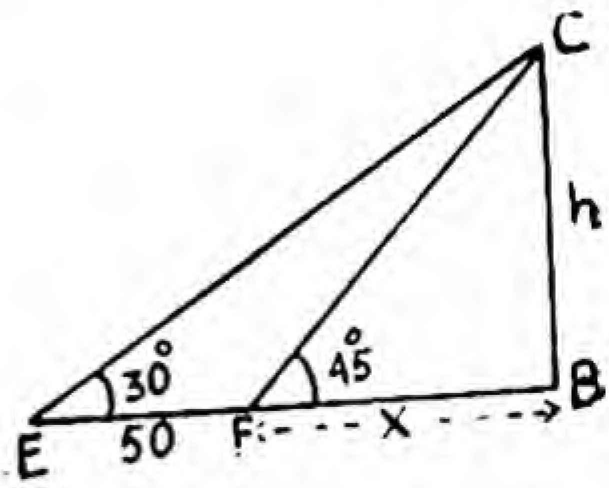
Now $\frac{BC}{AB}$ =tan $\hat{BAC}=1$

or $\frac{600}{x}$=tan 45=1

or $\frac{600}{x}=1$ or x=600 ft.

Ex. 3. A man standing on the bank of a river observes angle of elevation subtended by a tree top on the opposite bank to be 45°. On retiring 5 metres the angle of elevation diminishes to 30° ; find the height of the tree and the breadth of the river.

Sol. Let C be the top of the tree and F and E the two points of observation.



Let BC$=h$ (*the height of the tree*)

FB$=x$ (*the breadth of the river*)

Now given $\hat{CEF}=30°$; $\hat{CFB}=45°$

$$\frac{CB}{FB}=\tan \widehat{CFB} \text{ or } \frac{h}{x}=\tan 45=1$$

$$\text{or } h=x \qquad \ldots\ldots (i)$$

$$\text{Again } \frac{BC}{EB}=\tan \widehat{BEC} \text{ or } \frac{h}{x+50}=\tan 30=\frac{1}{\sqrt{3}}.$$

$$\text{or } \sqrt{3}\ h=x+50 \qquad \ldots\ldots (ii)$$

Now $h=x$ ......(i)

$\sqrt{3}\ .\ h=x+50$ ......(ii)

Substituting the value of h in the (ii)

we have $\sqrt{3}\ .\ x=x+50$ or $x(\sqrt{3}-1)=50$

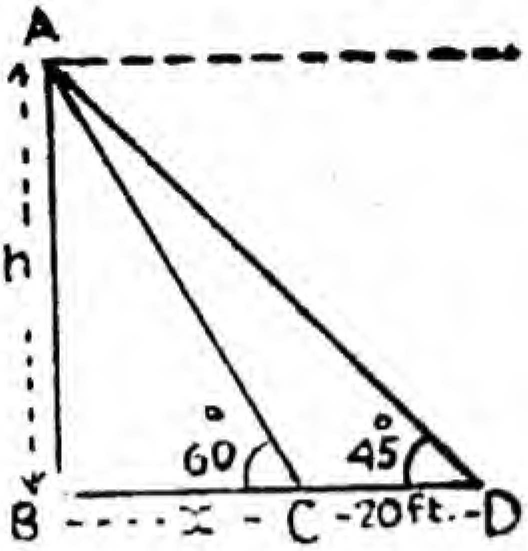
$$\text{or } x=\frac{50}{\sqrt{3}-1} \text{ metres.}$$

$$h=x=\frac{50}{\sqrt{3}-1} \text{ metres.}$$

Ex. 4. The angles of depression of two motor cars standing on a road and observed from the top of a tower are 45° and 60° respectively. If the cars and tower are in the same vertical plane and the cars 200 ft. apart, find the height of the tower.

(*K.U. 1952*)

(K.U. 1952)



Sol: Let AB be the tower h ft. high. C and D two motor cars 200 ft. apart,

$\widehat{ACB}=60°$; $\widehat{ADB}=45°$

Let $BC=x$ ft.

Now $\dfrac{AB}{BD}=\tan ADB$ or $\dfrac{h}{x+200}$

$=\tan 45°=1$

or $h=200+x \ldots(i)$

Again $\dfrac{AB}{BC}=\tan ACB$ or $\dfrac{h}{x}=\tan 60°=\sqrt{3}$

or $h=x\sqrt{3}. \quad \ldots(ii)$

$$h=200+x \qquad \ldots(i)$$

$$h=x\sqrt{3} \qquad \ldots(ii)$$

Substituting the value of x from (ii) in (i)

$$h=200+\frac{h}{\sqrt{3}} \text{ or } h\left(1-\frac{1}{\sqrt{3}}\right)=200$$

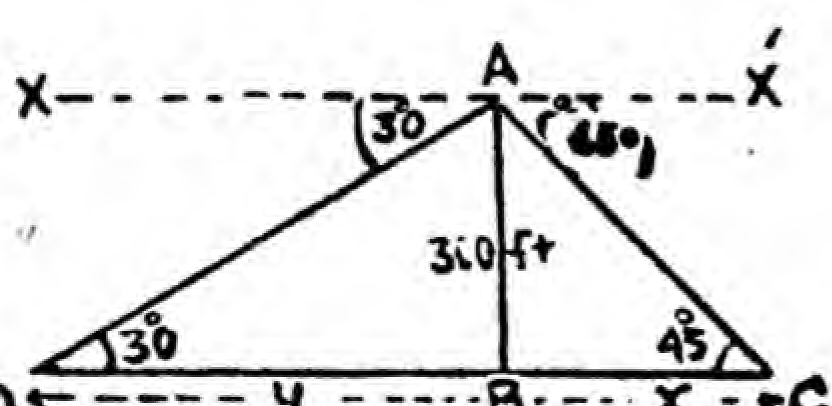
$$\text{or} \quad h=\frac{200}{1-\dfrac{1}{\sqrt{3}}}=\frac{200\sqrt{3}}{\sqrt{3}-1} \text{ ft.}$$

Ex. 5. From a lighthouse the angles of depression of two ships on opposite sides of the lighthouse are observed to be 30° and 45°. If the height of the lighthouse be 300 ft. Find the distance between the ships if the line joining them passes through the foot of the lighthouse. *(P.U. 1941)*

Sol. Let AB be the height of the tower=300 ft. Let C and D be the two ships. Then

(i) $\widehat{X'AC}\ (=\widehat{ACB})=45°$

(ii) $\widehat{XAD}(=\widehat{ADB})=30°$



Let BC $=x$ ft. ; DB $=y$ ft. (a) $\frac{AB}{BC}=\tan ACB$ or $\frac{300}{x}=\tan 45=1$

or $x=300$ ft....(i)

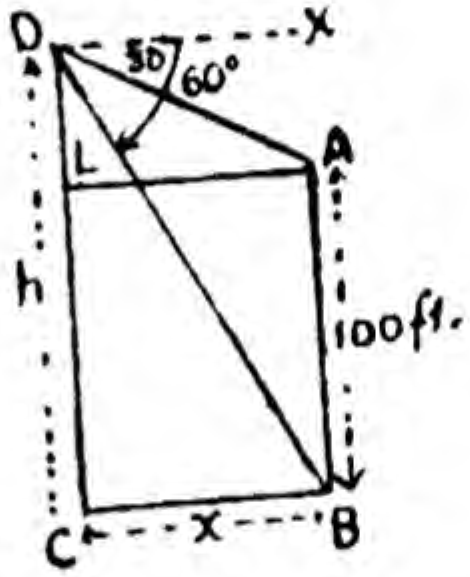
(b) $\frac{AB}{BD}=\tan A\widehat{D}B$ or $\frac{300}{y}=\tan 30=\frac{1}{\sqrt{3}}$

or $y=300\sqrt{3}$ ft.

Distance between the two ships DC

$=x+y=300+300\sqrt{3}$ or $300\,(1+\sqrt{3})$ ft.

Ex. 6. From the top of a tower the angles of depression of the top and the bottom of a building 100 ft. high are 30° and 60° respectively. Find the height of the tower and distance from the building.



Sol. Let the tower CD be h ft. high and the distance BC between the tower and the building AB be x ft.

We know (i) $\hat{XDA}\ (=\hat{DAL})=30°$

(ii) $\hat{XDB}\ (=\hat{DBC})=60°$

(iii) AB $=100$ ft.

Now (a) $\frac{DC}{BC}=\tan DBC$ or $\frac{h}{x}=\tan 60°=\sqrt{3}$ ......(i)

or $h=x\sqrt{3}$

(b) $\frac{DL}{AL}=\tan DAL$ or $\frac{h-100}{x}=\tan 30=\frac{1}{\sqrt{3}}$ ......(ii)

or $h\sqrt{3}-1000\sqrt{3}=x$

$h=x\sqrt{3}$ ......(i)

$h\sqrt{3}-100\sqrt{3}=x$ ......(ii)

Substituting the value of h from (i) into (ii)

$x\sqrt{3}.\sqrt{3}-100\sqrt{3}=x$

or $4x=100\sqrt{3}$ or $x=\frac{100\sqrt{3}}{4}=25\sqrt{3}$ ft.

$h=x\sqrt{2}=25\sqrt{3}x\sqrt{3}$

$=75$ ft.

EXERCISE I

1. A vertical flagstaff stands on a horizontal plane. From a point distant 150 ft. from its foot, the angle of elevation of top is $30°$; find the height of the flagstaff.

(H.S.B., Delhi)

2. A kite string is 150 yds. long and its angle af elevation is $60°$. Find the height of the kite above the ground.

3. From the top of a tower 125 ft. a man observes the angle of depression of a tree to be $30°$. Find the distance of the tree from the foot of the tower.

4. Find the altitude of the sun when the length of the shadow of a pole 30 ft. high is $30\sqrt{3}$.

5. A chimney 20 ft high standing on the top of a building, subtends an angle whose tangent is $\frac{1}{6}$ at a distance of 70 ft. from the foot of the building. Find the height of the building.

(H.S.E. Delhi 1953)

6. The upper part of a tree broken over by wind, makes an angle of $30°$ with the ground, and the distance from the root to the point where the top of the tree meets the ground is 30 ft.; what was the height of the tree. *(D. Qualifying 1951)*

7. A vertical post casts a shadow 20 ft. long when the altitude of the sun is $60°$. Find the length of the shadow when the altitude of the sun is $30°$.

8. The altitude of the top of a chimney is $30°$, approaching 200 ft. towards it, its magnitude becomes $45°$. Find the height of the chimney. *(K.U. 1951)*

9. A person standing on the bank of a river, observes that the angle subtended by a tree on the opposite bank is $60°$ when he retires 40 ft. from the bank he finds the angle to be $30°$; find the height of the tree and the breadth of the river.

(P.U. 1942 S)

10. The angles of elevation of the top of a tower observed by two observers standing on a road, on the opposite sides of the tower are $30°$ and $60°$ respectively. If the observers and the tower are in the same plane and are 500 ft. apart, find the height of the tower. *(K. U. 1953)*

11. From the top of a tower 100 ft. high, the angles of depression of two objects due north of the tower are 30° and 45°. Find the distance between the objects.

12. From a lighthouse the angles of depression of two ships on opposite sides on the lighthouse are observed to be 30° and 45°. If the height of the lighthouse be 300 ft., find the distance between the ships if the line joining them passes through the foot of the lighthouse. *(P.U. 1941)*

13. From the top of a cliff, 300 ft. high the top and bottom of a tower are observed to be 30° and 60° respectively. Find the height of the tower.

14. The angles of elevation of two points A and B on a vertical tower are 60° and 30 respectively from a point C on the ground. If AB = 100 ft. Find the height of A above the ground and the distance of the tower from C. *(Q. Delhi 1949)*

15. From the top of a tower 100 ft. high angle of elevation of a cloud is 30° and angle of depression of the image in lake is 60°. Find the height of the cloud.

16. If p is the length of the perpendicular from A to BC in a triangle ABC, prove that

$$p=\frac{a}{\text{Cot B}+\text{Cot C}}$$

(P.U. 1941)

17. A verticle tower stands on a horizontal plane and is surmounted by a vertical flagstaff of height h. At a point on the plane, the angle of elevation of the bottom of the flagstaff is α and that of the top of the flagstaff is β. Prove that the height of the tower is $\dfrac{h \tan \alpha}{\tan \beta-\tan \alpha}$ *(P.U. 1949)*

18. The angle of elevation of a tower from a point A due south of it is x and from a point B due east A is y. If AB $=l$, show that the height of the tower is given by $h^2\ (\text{Cot}^2 y-\text{Cot}^2 x)=l^2$ *(P.U. 1943)*

19. The angular elevation of a tower from a certain point is α ; at another point in the same horizontal plane and d feet nearer the tower, the elevation is $90^\circ - \alpha$. if h be the height of the tower above the horizontal plane, show that

$$h = \tfrac{1}{2}\, d \tan 2\alpha \text{ feet.}$$

20. AB is a tower standing on a horizontal plane, B being its foot. The elevations of A as observed from P due south and of Q due west of B are θ and φ respectively. If PQ is h feet, show that the height of the tower is

$$\frac{h}{\sqrt{\cot^2\theta + \cot^2\varphi}}$$

21. A lighthouse of height "a" feet is situated on the edge of a vertical cliff h feet high. From a boat the angle of elevation of the top of the lighthouse is α. When the boat has been moved x feet directly towards the lighthouse, the angles of elevation of the foot and the top of the lighthouse are α and β respectively. Prove that

$$a \tan \alpha = h\,(\tan \beta - \tan \alpha) = x \tan^2\alpha \qquad (K.U.\ 1962)$$

CHAPTER II

Relations between the Trigonometrical ratios of angles differing by multiple of right angles.

2.1 In the present chapter we shall discuss relations between Trigonometrical ratios of angles differing by multiple of rt. angles. It is very essential for the student to know such relations, and he is advised to understand these as thoroughly as possible.

2.2 **Functions of** $(-\theta)$

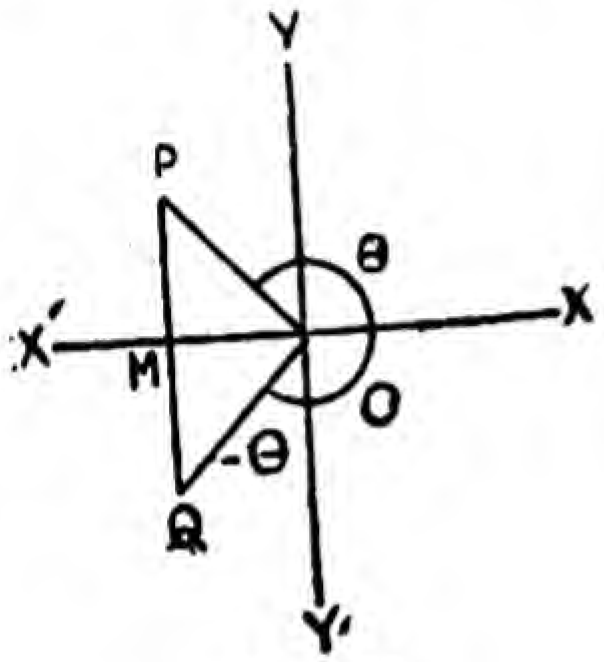
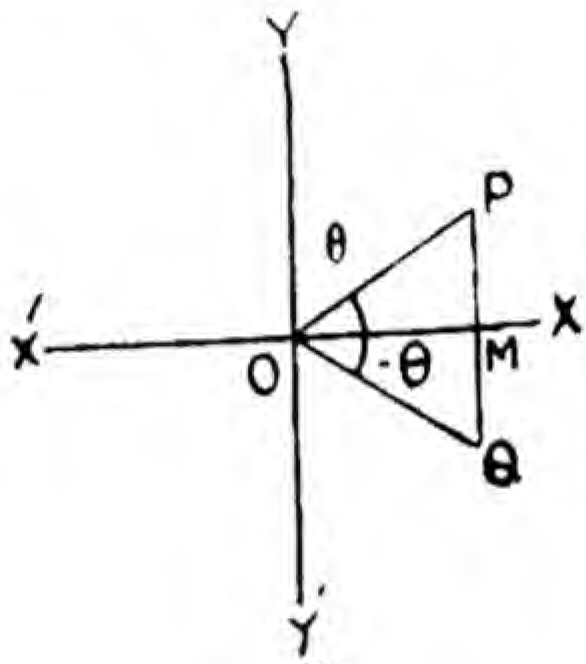


Fig. (ii)



(i)

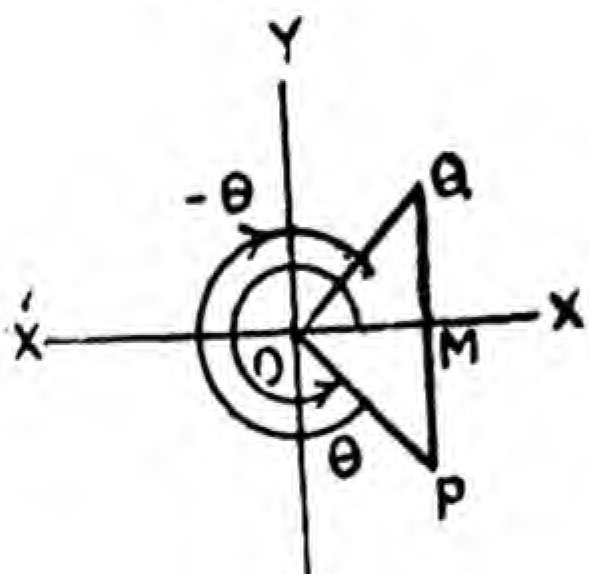


Fig. (iv)

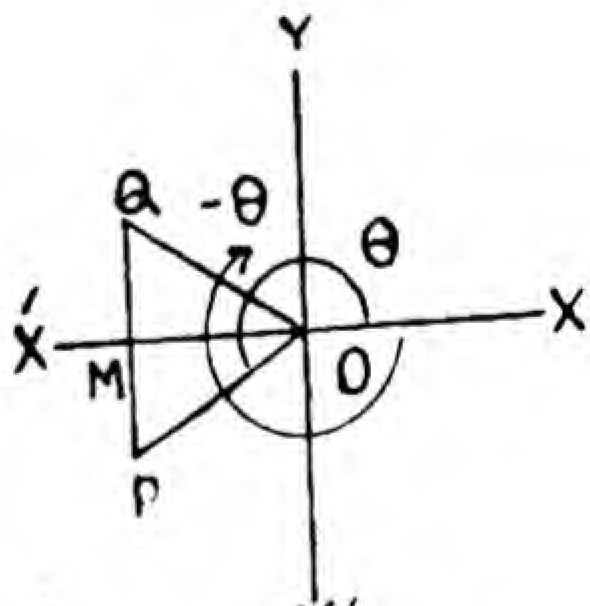


Fig. (iii)

Let the revolving line OP starting from its initial position OX trace out $\angle XOP=\theta$. Let the other revoloving line OQ $(=OP)$

starting from OX revolve in the opposite direction through O such that $\angle XOQ = -\theta$.

[$< XOQ$ is $-$ve because it has been traced in clock-wise direction.]

Draw PM$\perp$ or OX and produce it to meet OQ in Q.

Now $\triangle$s OMP, and OMQ are congruent ...(Why ?)

$\because$ QM$=-$MP ...($\because$ They have opposite signs)

OP$=$OQ ...(Construction)

and OM$=$OM

$$\therefore \quad \text{Sin } (-\theta) = \frac{QM}{OQ} = -\frac{MP}{OP} = -\text{Sin } \theta$$

$$\text{Cos } (-\theta) = \frac{OM}{OQ} = \frac{OM}{OP} = \text{Cos } \theta$$

$$\text{tan } (-\theta) = \frac{QM}{OM} = -\frac{PM}{OM} = -\text{tan } \theta$$

$$\text{Cot } (-\theta) = \frac{OM}{QM} = -\frac{OM}{PM} = -\text{Cot } \theta$$

$$\text{Sec } (-\theta) = \frac{OQ}{OM} = \frac{OP}{OM} = \text{Sec } \theta$$

$$\text{Cosec } (-\theta) = \frac{OQ}{QM} = -\frac{OP}{PM} = -\text{Cosec } \theta$$

Note :—(1) The angle $(-\theta)$ being in the (*iv*) quadrant, only *cosine* is positive, while all other ratios are negative.

Note :—(2) *How to draw the four figures.*

For the sake of convenience, take $\theta = 30°$ and then (adding 90° each time) 120°, 210°, and 300°. This will give us the position of OP in all the four quadrants.

Again, take $-\theta = -30°$, $-120°$, $-210°$, and $-300°$, which will give us the position of OQ in all quadrants.

The student is advised to have ample practice in drawing all the four figures.

2·3. Functions of $(90°-\theta)$

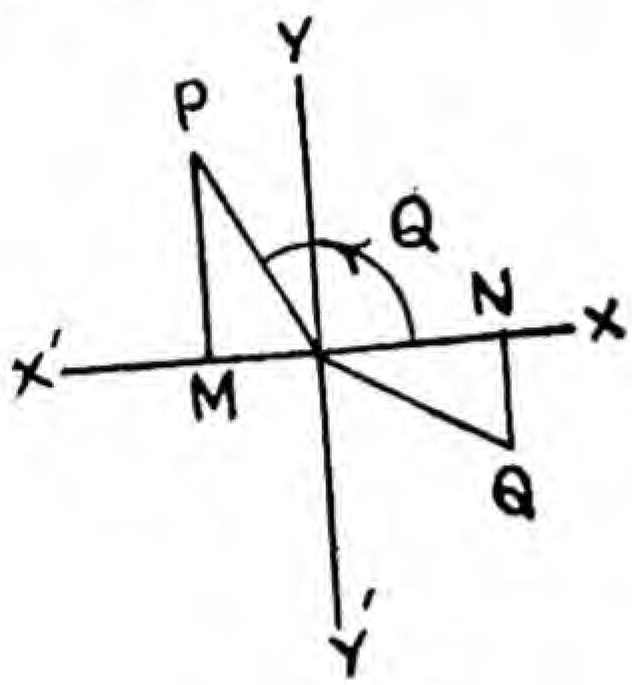


Fig. (ii)

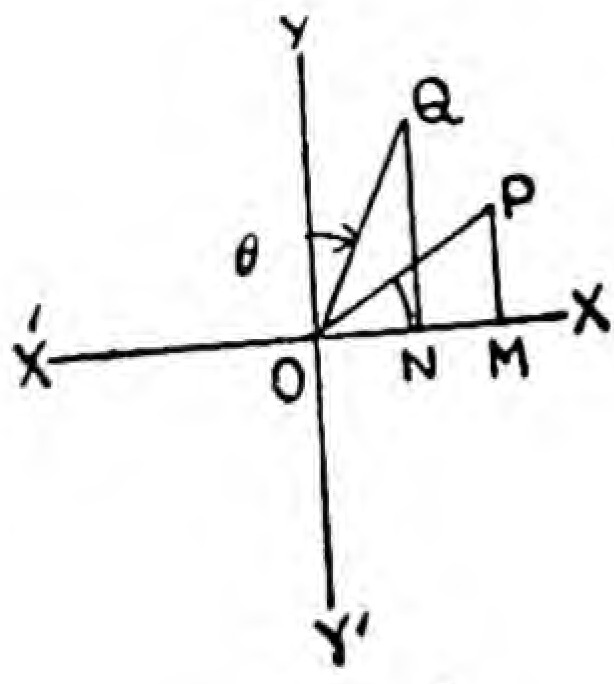


Fig. (i)

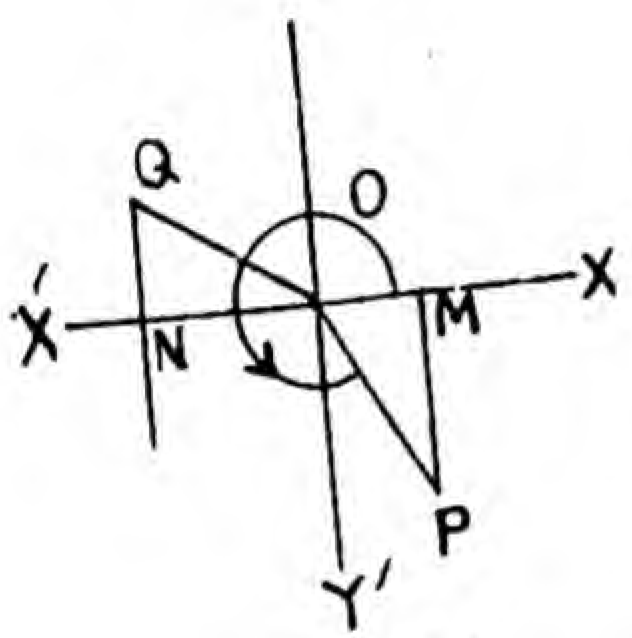


Fig. (iv)

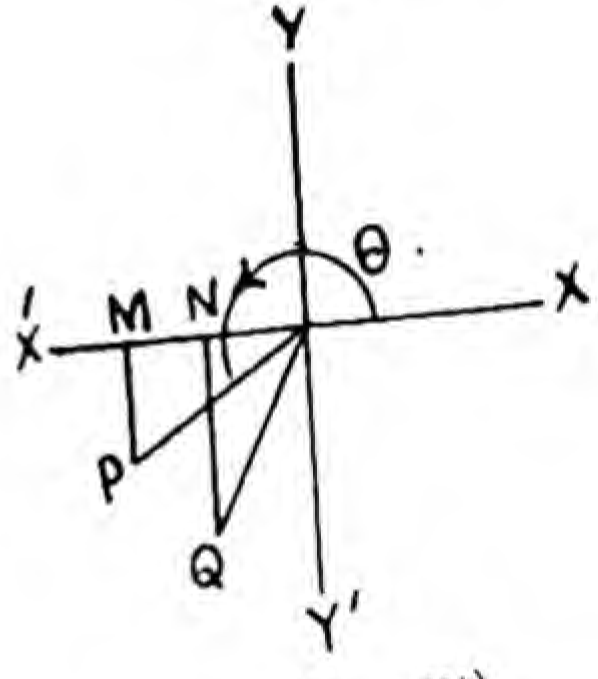


Fig. (iii)

Let the revolving line OP starting from OX trace out an angle $\angle XOP=\theta$.

Let another revolving line OQ $(=OP)$ start from OX trace out $\angle XOY=90°$. Let it then revolve back through θ, so that $\angle XOQ=90°-\theta$.

Draw PM and QN perpendiculars upon XOX'.

Now Δs OMP and ONQ are congruent ......(why ?)

$\therefore$ OM=NQ and MP=ON

$$\text{Now } \sin(90°-\theta)=\frac{NQ}{OQ}=\frac{OM}{OP}=\cos\theta$$

$$\cos(90°-\theta)=\frac{ON}{OQ}=\frac{MP}{OP}=\sin\theta$$

$$\tan\ (90^\circ-\theta)=\frac{NQ}{ON}=\frac{OM}{MP}=\text{Cot}\ \theta$$

$$\text{Cot}\ (90^\circ-\theta)=\frac{ON}{NQ}=\frac{MP}{OM}=\tan\ \theta$$

$$\text{Sec}\ (90^\circ-\theta)=\frac{OQ}{ON}=\frac{OP}{MP}=\text{Cosec}\ \theta$$

$$\text{Cosec}\ (90^\circ-\theta)=\frac{OQ}{NQ}=\frac{OP}{OM}=\text{Sec}\ \theta.$$

Note :—*How to draw the figures ?*

(*i*) To get the position of OP in all the four quadrants, take $\theta=30^\circ, 120^\circ, 210^\circ, 300^\circ$.

(*ii*) To get the position of OQ in all the four quadrants, take $90^\circ-\theta=60^\circ,\ -30^\circ,\ -120^\circ$ and -210°.

2.4 Functions of $(90^\circ+\theta)$

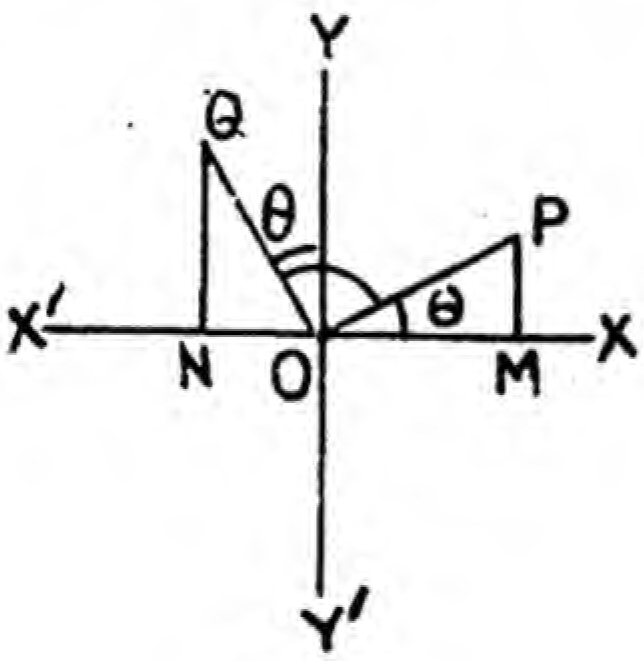


Fig. (i)

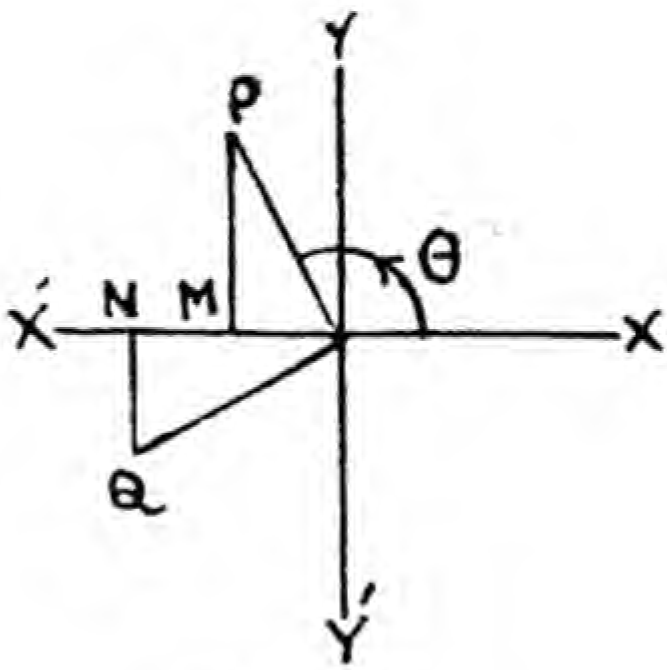
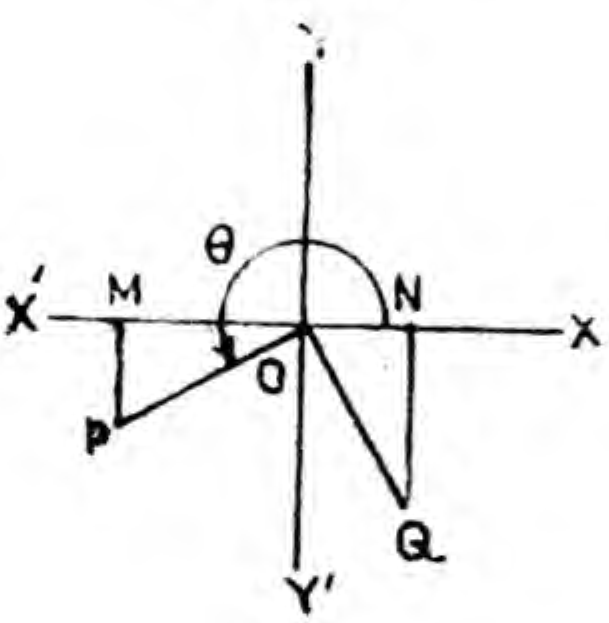
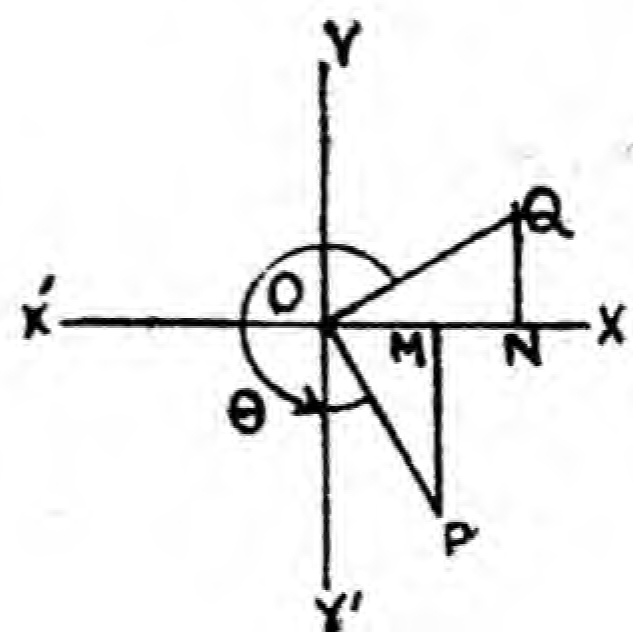


Fig. (ii)





Let the revolving line start from its initial position OX, trace out $\angle XOP=\theta$.

Let another revolving line OQ $(=OP)$ starting from OX trace out $\angle XOY=90°$, and then revolve further through θ, so that $\angle XOQ=90°+\theta$. Draw PM and QN $\perp$s upon XOX'

Now $\triangle$ OMP and ONQ are congruent ......(Why ?)

$\therefore$ OM=NQ and $-PM=ON$ (......$\because$ PM aud ON are opposite in sign)

$$\text{Now } \sin(90°+\theta)=\frac{NQ}{OQ}=\frac{OM}{OP}=\cos\theta$$

$$\cos(90°+\theta)=\frac{ON}{OQ}=\frac{PM}{OP}=-\sin\theta$$

$$\tan(90°+\theta)=\frac{NQ}{ON}=-\frac{OM}{PM}=-\cot\theta$$

$$\cot(90°+\theta)=\frac{ON}{NQ}=-\frac{PM}{OM}=-\tan\theta$$

$$\sec(90°+\theta)=\frac{OQ}{ON}=-\frac{OP}{PM}=-\operatorname{cosec}\theta$$

$$\operatorname{cosec}(90°+\theta)=\frac{OQ}{NQ}=\frac{OP}{OM}=\sec\theta$$

Note :—*How to draw the figures ?*

(i) To get the position of OP, take $\theta=30°$, $120°$, $210°$ and $300°$.

(ii) To get the position of OQ, take $90°+\theta=120°$, $210°$, $300°$, $390°$.

Important Rule :—*How to remember the results of articles 2.3 and 2.4.*

The functions of $90°-\theta$, $90°+\theta$ are changed into their Co–functions i.e., Sin into Cos, Cos into Sin, tan into Cot, Cot into tan, and so on. However, the function of $90°-\theta$ being in the first quadrant, all ratios are positive, whereas $90°+\theta$ being in the second quadrant, all ratios are negative except Sin and cosec.

The student is advised to understand this important rule thoroughly.

2.5 Functions of $(180^\circ - \theta)$

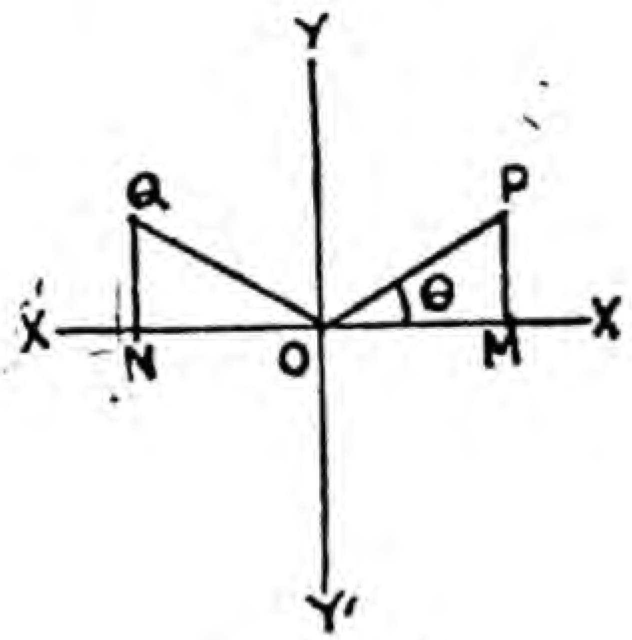


Fig. (i)

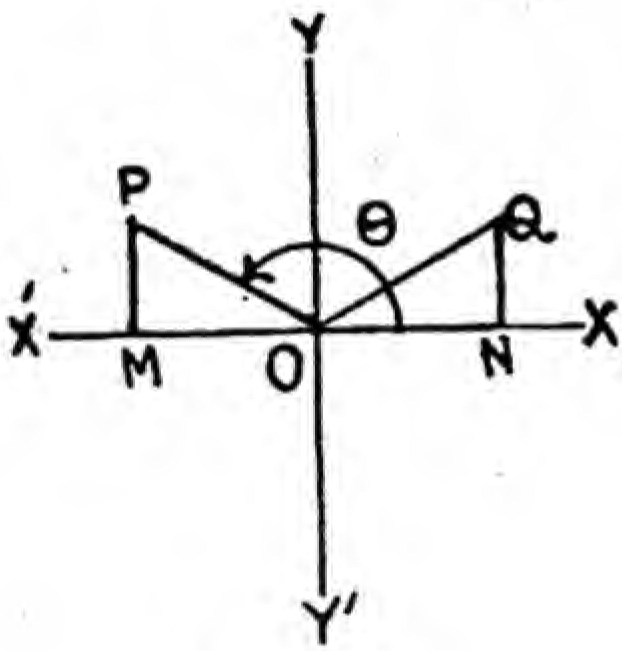


Fig (ii)

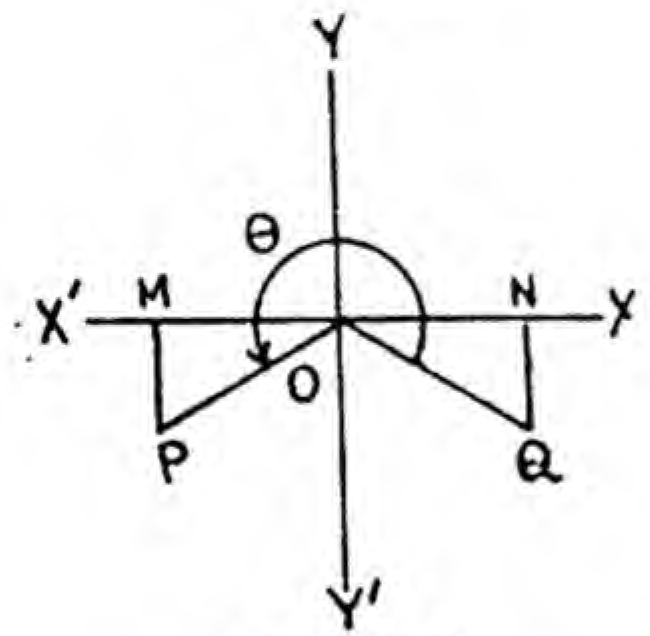


Fig. (iii)

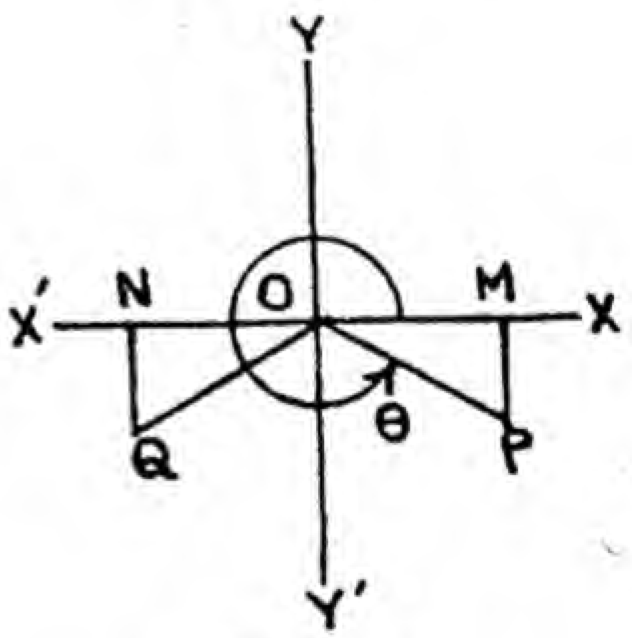


Fig. (iv)

Let the revolving line OP starting from OX trace out $\angle XOP = \theta$. Let another revolving line OQ $(= OP)$ starting from OX trace out $\angle XOX' = 180^\circ$, and let then revolve it back through θ, so that $\angle XOQ = 180^\circ - \theta$.

Draw PM and QN perpendiculars upon XOX'. $\triangle$s OMP and ONQ are cangruent ...(Why ?)

$\therefore$ $ON = -OM$ ...[equal but opposite in sign]

$QN = MP$

$$\text{Now Sin } (180^\circ-\theta)=\frac{NQ}{OQ}=+\frac{PM}{OP}=+\text{Sin }\theta$$

$$\text{Cos } (180^\circ-\theta)=\frac{ON}{OQ}=-\frac{OM}{OP}=-\text{Cos }\theta$$

$$\tan (180^\circ-\theta)=\frac{NQ}{ON}=-\frac{MP}{OM}=-\tan\theta$$

$$\text{Cot } (180^\circ-\theta)=\frac{ON}{NQ}=-\frac{OM}{MP}=-\text{Cot }\theta$$

$$\text{Sec } (180^\circ-\theta)=\frac{OQ}{ON}=-\frac{OP}{OM}=-\text{Sec }\theta$$

$$\text{Cosec } (180^\circ-\theta)=\frac{OQ}{NQ}=\frac{OP}{MP}=\text{Cosec }\theta$$

2.6 Functions of $(180^\circ+\theta)$

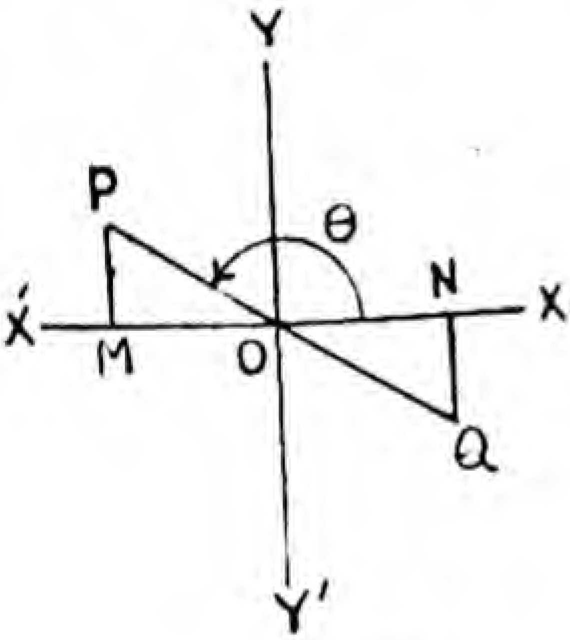


Fig. (ii)

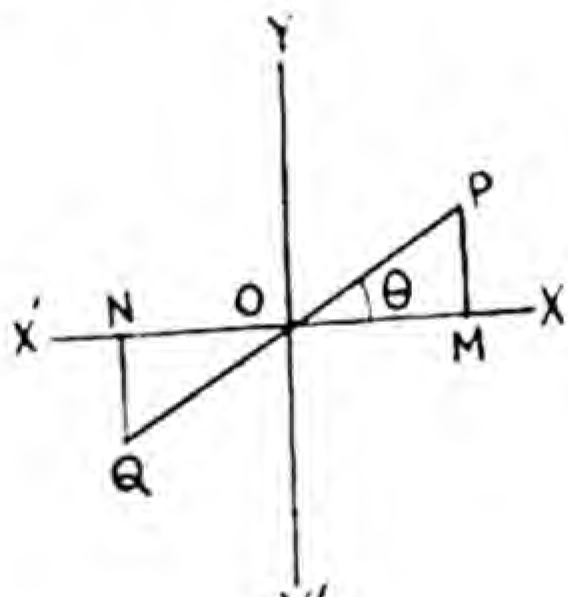


Fig. (i)

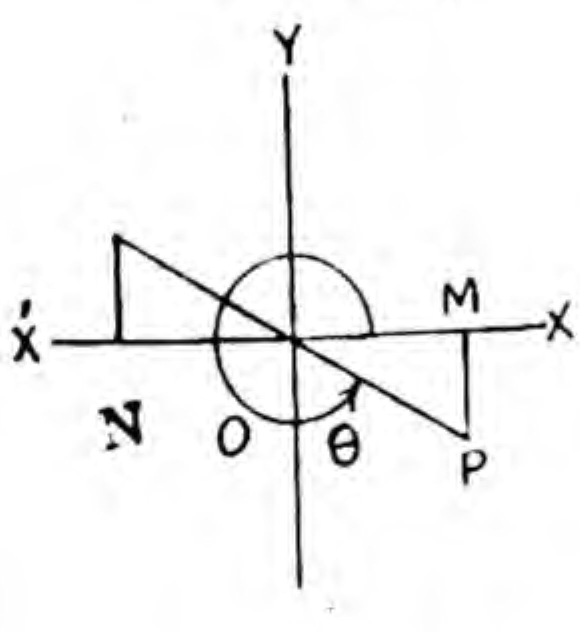


Fig. (iv)

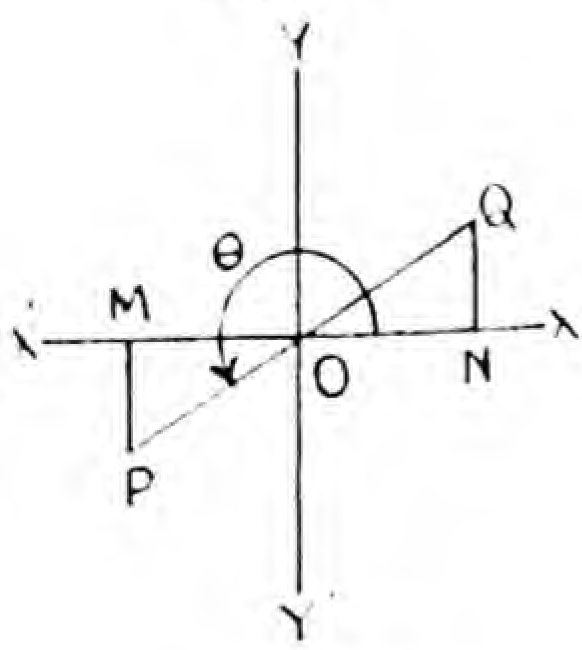


Fig. (iii)

Let the revolving line OP starting from OX trace out $\angle XOP=\theta$. Let another revolving line OQ $(=OP)$ starting from OX trace out $\angle XOX'=180°$. Let it then revolve further through θ, so that $\angle XOQ=180°+\theta$.

Draw PM and QN $\perp$s on XOX'

$\triangle$s OPM and OQN are congruent ...(why ?)

$\therefore$ ON $=-$OM

NQ $=-$PM

$$\text{Now Sin }(180°+\theta)=\frac{NQ}{OQ}=-\frac{PM}{OP}=-\text{Sin }\theta$$

$$\text{Cos }(180°+\theta)=\frac{ON}{OQ}=-\frac{OM}{OP}=-\text{Cos }\theta$$

$$\tan\ (180°+\theta)=\frac{NQ}{ON}=\frac{-PM}{-OM}=\frac{PM}{OM}=\tan\theta$$

$$\text{Cot }(180°+\theta)=\frac{ON}{NQ}=\frac{-OM}{-PM}=\frac{OM}{PM}=\text{Cot }\theta$$

$$\text{Sec }(180°+\theta)=\frac{OQ}{ON}=-\frac{OP}{OM}=-\text{Sec }\theta$$

$$\text{Cosec }(180°+\theta)=\frac{OQ}{NQ}=-\frac{OP}{PM}=-\text{Cosec }\theta$$

Note :—*How to remember the results of articles 2·5 and 2.6.*

The functions of $180°-\theta$ and $180°+\theta$ remain the same, but $180°-\theta$ lying in the second quadrant. only Sin and Cosec are positive, whereas $180°+\theta$ lying in the 3rd quadrant, only tan and Cot are positive.

2.7 A VERY IMPORTANT RULE

The student is required to understand the following rule thoroughly. This will enable him to remember the results obtained in articles 2·2—2·6.

1. (*a*) When an angle is an odd multiple of 90°, *i.e.*, 90°, 270°, *etc.*, the functions change into their co—functions, and the sign is determined with the help of the quadrant in which the angle lies, for instance, Cos $(270°-\theta)=-\text{Sin }\theta$ (*i*)

$$\tan\ (270^\circ-\theta)=+\text{Cot}\ \theta \qquad (ii)$$
$$\text{Cos}\ (90^\circ+\theta)=-\text{Sin}\ \theta \qquad (iii)$$

and so on.

(*i*) Here the angle lies in 3rd quadrant.

(*ii*) Here the angle lies in 3rd quadrant.

(*iii*) Here the angle lies in the 2nd quadrant.

(*b*) When the angle is an even multiple of 90°, *i.e.*, 180°, 360°, *etc.*, the functions remain the same, and their signs are determined by the quadrants in which these angles lie.

For example, $\tan\ (180^\circ+\theta)=\tan\ \theta \qquad ...(i)$

$$\text{Cot}\ (360^\circ-\theta)=-\text{Cot}\ \theta \qquad ...(ii)$$
$$\text{Sin}\ (360^\circ+\theta)=\text{Sin}\ \theta \qquad ...(iii)$$

and so on.

(*i*) Here the angle lies in the 3rd quadrant

(*ii*) Here the angle lies in the 4th quadrant

(*iii*) Here the angle lies in the Ist quadrant.

II. When θ, is changed into $-\theta$ the functions remain the same, but are all negative except Cos and Sec, as the sign is determined by the quadrant in which the angle lies.

For example, $\text{Sin}\ (-\theta)=-\text{Sin}\ \theta$, $\text{Cos}\ (-\theta)=\text{Cos}\ \theta$ etc. $\left\{\because (-\theta) \text{ lies in the 4th quadrant}\right\}$

2.8 Periodic Functions

Def. A function $f(x)$ is said to be *periodic* if its value remains unaltered when x is changed into $x+a$, *i.e.*, $f(x)=f(x+a)$ for all values of x, a is said to be the period of the function.

2.8.1 *Periods of Sin θ, Cos θ, and tan θ*

We know that $\text{Sin}\ (\theta+2\pi)=\text{Sin}\ \theta$, $\text{Cos}\ (\theta+2\pi)=\text{Cos}\ \theta$ $\left\{\text{Rule I } (b) \text{ under Article 2·7}\right\}$

Thus we see that the values Sin θ, Cos θ remain unchanged when 2π is added to 2θ. Hence Sin θ and Cos θ are periodic functions and their period is 2π in each case.

Again $\tan(\theta+\pi)=\tan\theta$ [Rule I (*b*) under Article 2·7]

Which shows that tan θ is also periodic with π as its period.

Solved Examples

Ex. 1. *Find the values of* :—

(*i*) *Cot* 570° (*ii*) *Cos* 720° (*iii*) *tan* (-1215°)

Sol. (*i*) $\text{Cot } 570^\circ = \text{Cot } (6\times 90^\circ + 30^\circ) = +\text{Cot } 30^\circ) = \sqrt{3}$

(Here the angle 30° is an even multiple of 90° and lies in the 3rd quadrant)

(*ii*) $\text{Cos } 720^\circ = \text{Cos } (8\times 90^\circ + 0^\circ) = +\text{Cos } 0^\circ = 1$

(Here the angle 0° is an even multiple of 90°, and lies in the 1st quadrant)

(*iii*) $\tan(-1215^\circ) = -\tan 1215^\circ = -\tan(13\times 90^\circ + 45^\circ)$
$= -(-\text{Cot } 45^\circ) = \text{Cot } 45^\circ = 1$

(Here the angle 45° is an odd multiple of 90°

$\therefore$ $\tan(13\times 90^\circ + 45^\circ)$ will change into its Co-function *i.e.*, Cot 45° with $-$ve sign because the angle $13\times 90^\circ + 45^\circ$ lies in the 2nd quadrant. Also $\tan(-\theta) = -\tan\theta$

$\therefore$ $\tan(-1215^\circ) = -\tan 1215^\circ$)

Ex. 2. *Show that.*

$\text{Sin } 420^\circ \text{ Cos } 390^\circ + \text{Cos } (-660^\circ) \text{ Sin } (-330^\circ) = 1$

Sol. L.H.S. $= \text{Sin } (4\times 90^\circ + 60^\circ) \text{ Cos } (4\times 90^\circ + 30^\circ)$
$\text{Cos } (7\times 90^\circ + 30^\circ) \text{ Sin } (4\times 90^\circ - 30^\circ)$
$[\because \text{Sin } (-\theta) = -\text{Sin } \theta \text{ and Cos } (-\theta) = \text{Cos } \theta]$

$= \text{Sin } 60^\circ \text{ Cos } 30^\circ - \text{Sin } 30^\circ (-\text{Sin } 30^\circ)$ (why so ?)

$= \text{Sin } 60^\circ \text{ Cos } 30^\circ + \text{Sin } 30^\circ \text{ Sin } 30^\circ$

$= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4} + \frac{1}{4} = 1 = \text{R. H.S.}$

Ex. 3. *If A, B, C are the angles of a triangle prove that :—*

(*i*) $\text{Sin}(A+B)=\text{Sin } C$ (*ii*) $\text{Cos}\frac{A+B}{2}=\text{Sin}\frac{C}{2}$

Sol. (*i*) $A+B+C=180°$

$$A+B=180°-C$$

$$\therefore \text{Sin}(A+B)=\text{Sin}(180°-C)=\text{Sin } C$$

(*ii*) $\frac{A}{2}+\frac{B}{2}+\frac{C}{2}=90°$

$$\frac{A+B}{2}=90°-\frac{C}{2}$$

$$\therefore \text{Cos}\frac{A+B}{2}=\text{Cos}\left(90°-\frac{C}{2}\right)=\text{Sin}\frac{C}{2}$$

Ex. 4. *Prove that :—*

$$\frac{\text{Cos }\theta}{\text{Sin }(90°+\theta)}+\frac{\text{Sin }(-\theta)}{\text{Sin }(180°+\theta)}-\frac{\tan(90°+\theta)}{\text{Cot }\theta}=3$$

Sol. L.H.S. $=\frac{\text{Cos }\theta}{\text{Sin }(90°+\theta)}+\frac{\text{Sin }(-\theta)}{\text{Sin }(180+\theta)}-\frac{\tan(90°+\theta)}{\text{Cot }\theta}$

$$=\frac{\text{Cos }\theta}{\text{Cos }\theta}+\frac{(-\text{Sin }\theta)}{(-\text{Sin }\theta)}-\frac{(-\text{Cot }\theta)}{\text{Cot }\theta}$$

$$=\frac{\text{Cos }\theta}{\text{Cos }\theta}+\frac{\text{Sin }\theta}{\text{Sin }\theta}+\frac{\text{Cot }\theta}{\text{Cot }\theta}=1+1+1$$

$$=3=\text{R.H.S.}$$

Ex. 5. *Show that* $\sin(n\pi+\theta)=\sin\theta$ or $-\sin\theta$, *according as n is even or odd.*

Sol. Case (*i*) When n is even.

Let $n=2m$ (say)

$\therefore \text{Sin}(n\pi+\theta)=\text{Sin}(2m\pi+\theta)=\text{Sin }\theta$

($\because$ this angle lies in the first quadrant for all m)

Case (*ii*) When n is odd.

Let $n = 2\,m+1$ (Say)

$$\therefore \sin(n\pi+\theta) = \sin[(2\,m+1)\pi+\theta]$$
$$= \sin(2m\pi+\pi+\theta)$$
$$= \sin(\pi+\theta) \qquad \text{...by case } (i)$$
$$= -\sin\theta$$

EXERCISE II

1. Prove that (*i*) $\cos 1385^\circ = \sin 35^\circ$

   (*ii*) $\tan(-965^\circ) = -\cot 25^\circ$

   (*iii*) $\sec(990^\circ - \theta) = -\operatorname{cosec}\theta$

2. Evaluate :—

   (*i*) $\cos\theta + \cos\left(\frac{\pi}{2}+\theta\right) + \cos(\pi+\theta) + \cos\left(\frac{3\pi}{2}+\theta\right)$

   (*ii*) $\sin^2\left(\frac{3\pi}{2}+\theta\right) + \cos^2\left(\frac{3\pi}{2}-\theta\right)$

   (*iii*) $\sec^2\left(\frac{7\pi}{2}-\theta\right) - \tan^2\left(\theta - \frac{9\pi}{2}\right)$

3. Prove that :—

   (*i*) $\sin 420^\circ \cos 390^\circ + \cos(-660^\circ)\sin(-330^\circ)$ 1

   (*ii*) $\sin 600^\circ \cos 330^\circ + \cos 120^\circ \sin 150^\circ = -1$

   (*iii*) $\sin^2 36^\circ - \sin^2 18^\circ = \sin^2 72^\circ - \sin^2 54^\circ$

   (*iv*) $\tan 225^\circ \cot 405^\circ + \tan 405^\circ \cot 675^\circ = 0$

4. Prove geometrically that :—

   (*i*) $\cos 150^\circ = -\cos 30^\circ$

   (*ii*) $\tan 225^\circ = \tan 45^\circ$

5. Show that :—

   (*i*) $\sin(180^\circ + A) = -\sin A$

   (*ii*) $\sec(270^\circ + A) = \operatorname{cosec} A$

   (*iii*) $\sec(270^\circ - A) = -\operatorname{cosec} A$

   (*iv*) $\cot(270^\circ + A) = -\tan A$

6. Simplify :—

(*i*) $\sin(180°+A)\ \text{cosec}\ (90°-A)$

(*ii*) $\tan(180°-A)\ \text{cosec}\ (180°+A)\sin(90°+A)$

(*iii*) $\dfrac{\cos(90°+\theta)\sec(-\theta)\tan(180°-\theta)}{\sec(360°-\theta)\sin(180°+\theta)\cot(90°-\theta)}$

(*iv*) $\dfrac{\sin(180°+\theta)\cos(270°-\theta)}{\sin(180°-\theta)\cos(270°+\theta)}$

(*v*) $\dfrac{\cos\theta}{\sin(90°-\theta)}+\dfrac{\sin(-\theta)}{\sin(180°+\theta)}+\dfrac{\tan(90°-\theta)}{\cot\theta}$

7. A, B, C, D are the angles of a quadrilateral, prove that

(*i*) $\sin(A+B)+\sin(C+D)=0$

(*ii*) $\cos(A+B)=\cos(C+D)$

8. A quadrilateral ABCD is inscribed in a circle. Show that.

(*i*) $\sin A=\sin C$ (*ii*) $\cos B+\cos D=0$

and (*iii*) $\cos A+\cos B+\cos C+\cos D=0$

9. Find x from the equation

$\text{cosec}\ (90°+A)+x\cos A\cot(90°+A)=\sin(90°+A)$

10. Show that *in general*, $\cos(m\pi+\theta)=(-1)^m\cos\theta$.

CHAPTER III

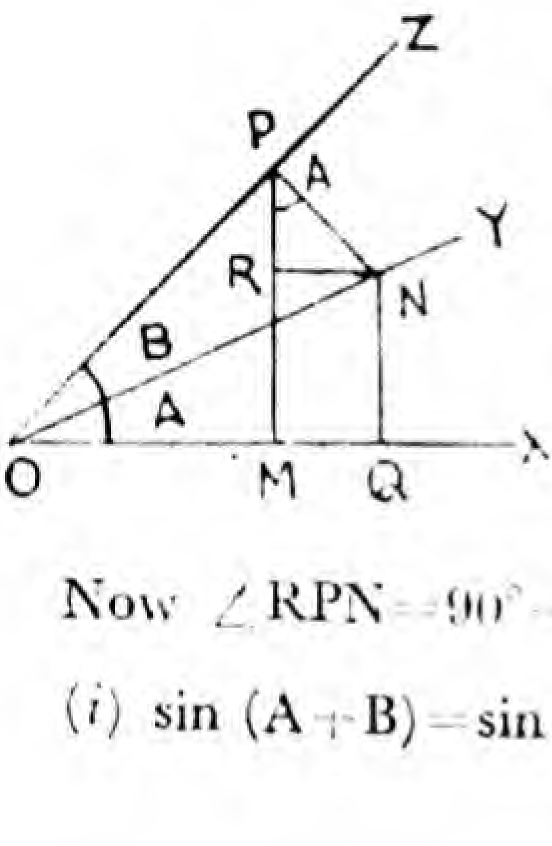
Addition And Subtraction Formulae

3·1. To prove *geometrically* that :—

(i) $\sin(A+B)=\sin A\cos B+\cos A\sin B$

(ii) $\cos(A+B)=\cos A\cos B-\sin A\sin B$

and (iii) $\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}$



Let the revolving line starting from its initial position OX, trace out an angle XOY $=$ A. Let it further trace out an angle YOZ$=$B, so that the angle XOZ $=$ A$+$B.

Take any point P on OZ and draw PM and PN perpendiculars on OX and OY respectively; from N draw NQ and NR perpendiculars on OX and PM respectively.

Now $\angle RPN=90°-\angle RNP=\angle RNO=\angle NOQ=A$

(i) $\sin(A+B)=\sin XOZ=\dfrac{MP}{OP}=\dfrac{MR+RP}{OP}=\dfrac{QN+RP}{OP}$

$$=\frac{QN}{OP}+\frac{RP}{OP}$$

$$=\frac{QN}{ON}\cdot\frac{ON}{OP}+\frac{RP}{NP}\cdot\frac{NP}{OP}$$

[Under QN write the hypotenuse of the rt. $\angle$d $\triangle$ of which QN is a side. Similarly, under RP write the hypotenuse of the rt. $\angle$d $\triangle$ of which RN is a side.]

$=\sin A\cos B+\cos A\sin B$ $\quad$ ($\because$ $\angle RPN=A$)

(*ii*) $\cos (A+B) = \cos XOZ = \frac{OM}{OP} = \frac{OQ-MQ}{OP} = \frac{OQ-RN}{OP}$

$$= \frac{OQ}{OP} - \frac{RN}{OP} = \frac{OQ}{ON} \cdot \frac{ON}{OP} - \frac{RN}{NP} \cdot \frac{NP}{OP}$$

[Under OQ write the hypotenuse of the rt. $\angle$d $\triangle$ of which OQ is a side. Under RN write the hypotenuse of the rt. $\angle$d$\triangle$ of which RN is a side.]

$= \cos A . \cos B - \sin A . \sin B$

(*iii*) $\tan (A+B) = \tan XOZ = \frac{MP}{OM} = \frac{MR+RP}{OQ-MQ} = \frac{QN+RP}{OQ-RN}$

$$= \frac{\frac{QN}{OQ} + \frac{RP}{OQ}}{1 - \frac{RN}{OQ}}$$ (Divide the num. and the Denom. by OQ)

$$= \frac{\tan A + \frac{RP}{OQ}}{1 - \frac{RN}{RP} \cdot \frac{RP}{OQ}}$$

But $\frac{RP}{OQ} = \tan A$, and from two similar $\triangle$s RPN and QON, $\frac{RP}{OQ} = \frac{NP}{ON} = \tan B$

Hence $\tan (A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

Or, we have :—

$$\tan (A=B) = \frac{\sin (A+B)}{\cos (A+B)} = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$$

$$= \frac{\frac{\sin A \cos B}{\cos A \cos B} + \frac{\cos A \sin B}{\cos A \cos B}}{\frac{\cos A \cos B}{\cos A \cos B} - \frac{\sin A \sin B}{\cos A \cos B}}$$

$$= \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

Caution : [Sin $(A+B)$ is *never equal* to $\sin A+\sin B$. Similarly, $\cos(A+B) \neq \cos A+\cos B$ and $\tan(A+B) \neq \tan A+\tan B$]

Cor. Prove that $\cot(A+B)=\dfrac{\cot A \cot B-1}{\cot A+\cot B}$

First method : $\text{Cot}(A+B)=\dfrac{\cos(A+B)}{\sin(A+B)}$

$$=\frac{\cos A\cos B-\sin A\sin B}{\sin A\cos B+\cos A\sin B}$$

$$=\frac{\dfrac{\cos A\cos B}{\sin A\sin B}-\dfrac{\sin A\sin B}{\sin A\sin B}}{\dfrac{\sin A\cos B}{\sin A\sin B}+\dfrac{\cos A\sin B}{\sin A\sin B}}$$

$$=\frac{\cot A\cot B-1}{\cot B+\cot A}$$

$$=\frac{\cot A\cot B-1}{\cot A+\cot B}$$

Second Method : We know $\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}$

$$\therefore \cot(A+B)=\frac{1}{\tan(A+B)}=\frac{1-\tan A\tan B}{\tan A+\tan B}$$

$$=\frac{1-\dfrac{1}{\cot A}\cdot\dfrac{1}{\cot B}}{\dfrac{1}{\cot A}+\dfrac{1}{\cot B}}$$

$$=\frac{\dfrac{\text{Cot A. Cot B}-1}{\text{Cot A Cot B}}}{\dfrac{\text{Cot B}+\text{Cot A}}{\text{Cot A. Cot B}}}=\frac{\text{Cot A Cot B}-1}{\text{Cot A}+\text{Cot B}}$$

Note : *The student is advised to commit this formula to memory, as he has to make frequent use of it in the forthcoming chapters.*

3·2 Addition formula for more than two angles.

To prove that :—

(*i*) $\sin(A+B+C)=\sin A \cos B \cos C+\sin B \cos C \cos A+\sin C \cos A \cos B-\sin A \sin B \sin C$

Here we have $\sin(A+B+C)=\sin[(A+B)+C]$

$=\sin(A+B)\cos C+\cos(A+B)\sin C$

$=(\sin A \cos B+\cos A \sin B)\cos C+(\cos A \cos B-\sin A \sin B)\sin C$

$=\sin A \cos B \cos C+\sin B \cos A \cos C+\sin C \cos A \cos B-\sin A \sin B \sin C$

(*ii*) $\cos(A+B+C)=\cos A \cos B \cos C-\cos A \sin B \sin C-\cos B \sin C \sin A-\cos C \sin A \sin B.$

Here $\cos(A+B+C)=\cos[(A+B)+C]=\cos(A+B)\cos C-\sin(A+B)\sin C$

$=(\cos A \cos B-\sin A \sin B)\cos C-(\sin A \cos B+\cos A \sin B)\sin C$

$=\cos A \cos B \cos C-\cos A \sin B \sin C-\cos B \sin C \sin A-\cos C \sin A \sin B$

(*iii*) $\tan(A+B+C)$

$$=\frac{\tan A+\tan B+\tan C-\tan A \tan B \tan C}{1-\tan A \tan B-\tan B \tan C-\tan C \tan A}$$

Here we have $\tan(A+B+C)=\tan[(A+B)+C]$

$$=\frac{\tan(A+B)+\tan C}{1-\tan(A+B)\tan C}$$

$$=\frac{\dfrac{\tan A+\tan B}{1-\tan A \tan B}+\tan C}{1-\dfrac{\tan A+\tan B}{1-\tan A \tan B}\tan C}$$

$$=\frac{\tan A+\tan B+\tan C-\tan A \tan B \tan C}{1-\tan A \tan B-\tan C \tan A-\tan B \tan C}$$

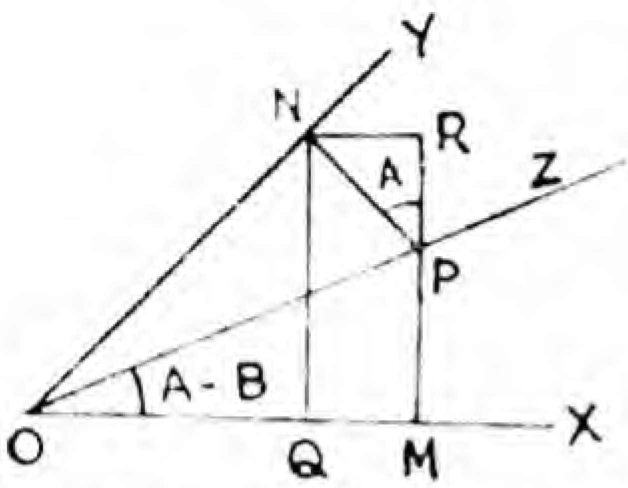
Note : *The student is advised to memorize the last formula.*

3·3 To prove *geometrically* that :—

(*i*) $\sin(A-B)=\sin A\cos B-\cos A\sin B.$

(*ii*) $\cos(A-B)=\cos A\cos B+\sin A\sin B$

and (*iii*) $\tan(A-B)=\dfrac{\tan A-\tan B}{1+\tan A\tan B}$



Let the revolving line, starting from its initial position OX, trace out an angle $XOY=A$. Let it then revolve back so as to trace an angle $YOZ=B$, so that the angle $XOZ=A-B$.

From any point P in OZ, draw PM and PN perpendiculars on OX and OY respectively. From N draw NQ and NR perpendiculars on OX and MP produced.

Then $\angle XOY=\angle RNY=90^\circ-\angle RNP=\angle RPN=A$

Now (*i*) $\sin(A-B)=\sin\angle XOZ=\dfrac{MP}{OP}=\dfrac{MR-PR}{OP}$

$$=\frac{QN-PR}{OP}=\frac{QN}{OP}-\frac{PR}{OP}$$

$$=\frac{QN}{ON}\cdot\frac{ON}{OP}-\frac{PR}{NP}\cdot\frac{NP}{OP}$$

[Under QN write the hypotenuse of the rt. ∠d △ of which QN is a side, and under PR write the hypotenuse of the rt. ∠d △ of which PR is a side.

$$=\sin A.\cos B-\cos A.\sin B$$

(*ii*) $\cos(A-B)=\cos\angle XOZ=\dfrac{OM}{OP}=\dfrac{OQ+QM}{OP}$

$$=\frac{OQ+NR}{OP}$$

$$=\frac{OQ}{OP}+\frac{NR}{OP}=\frac{OQ}{ON}\cdot\frac{ON}{OP}$$

$$+\frac{NR}{NP}\cdot\frac{NP}{OP}$$

[Under OQ write the hypotenuse of the rt. $\angle$d $\triangle$ of which OQ is a side, and under NR write the hypotenuse of the rt. $\angle$d$\triangle$ of which NR is a side].

$$=\text{Cos A Cos B}+\text{Sin A . Sin B}$$

(*iii*) $\tan(A-B)=\tan\angle XOZ=\frac{MP}{OM}=\frac{MR-PR}{OQ+QM}$

$$=\frac{QN-PR}{OQ+NR}=\frac{\frac{QN}{OQ}-\frac{PR}{OQ}}{1+\frac{NR}{OQ}}$$

$$=\frac{\tan A-\frac{PR}{OQ}}{1+\frac{NR}{PR}\cdot\frac{PR}{OQ}}$$

$$=\frac{\tan A-\frac{PR}{OQ}}{1+\tan A.\frac{PR}{OQ}}$$

But from the similar triangles QON and RPN, we have

$$\frac{PR}{OQ}=\frac{PN}{ON}=\tan B$$

Hence $\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}$

Or, we have :—

$$\tan(A-B)=\frac{\text{Sin}(A-B)}{\text{Cos}(A-B)}=\frac{\text{Sin A Cos B}-\text{Cos A Sin B}}{\text{Cos A Cos B}+\text{Sin A Sin B}}$$

$$=\frac{\frac{\text{Sin A Cos B}}{\text{Cos A Cos B}}-\frac{\text{Cos A Sin B}}{\text{Cos A Cos B}}}{\frac{\text{Cos A Cos B}}{\text{Cos A Cos B}}+\frac{\text{Sin A Sin B}}{\text{Cos A Cos B}}}=\frac{\tan A-\tan B}{1+\tan A\tan B}$$

Cor. To prove that $\text{Cot}(A-B)=\frac{\text{Cot A Cot B}+1}{\text{Cot A}-\text{Cot B}}$

(The proof of this cor is left to the student as an exercise)

Note :—*While proving the addition and subtraction formulae, we have drawn figures for the cases where A, B, A+B and A−B are all acute angles. But the same method can be extended to cases where there are obtuse angles as well.*

3.3.1 The three results obtained in article 3.3 can also be obtained by the method given below, but this will not be the geometrical method.

(*i*) To prove that Sin (A−B)=Sin A Cos B−Cos A Sin B. We have Sin (A+B)=Sin A Cos B+Cos A Sin B Put B=—B.

$$\therefore \text{Sin } (A-B) = \text{Sin } A \text{ Cos } (-B) + \text{Cos } A \text{ Sin } (-B)$$
$$= \text{Sin } A \text{ Cos } B - \text{Cos } A \text{ Sin } B$$

[∵Cos (−B)=Cos B and Sin (—B)=—Sin B.]

Similarly, we can prove the other two theorems also.

3.4 A Standard Result

To prove that $\tan\left(\frac{\pi}{4}+\theta\right)=\frac{1+\tan\theta}{1-\tan\theta}$

$$\text{L. H. S.} = \tan\left(\frac{\pi}{4}+\theta\right) = \frac{\tan\frac{\pi}{4}+\tan\theta}{1-\tan\frac{\pi}{4}\tan\theta}$$

$$= \frac{1+\tan\theta}{1-\tan\theta} \quad \left[\because \tan\frac{\pi}{4}=1\right]$$

Similarly, we can prove that

$$\tan\left(\frac{\pi}{4}-\theta\right)=\frac{1-\tan\theta}{1+\tan\theta}$$

The student is advised to commit both these results to memory.

Solved Examples

Ex. 1 Find the values of Sin 75°, Cos 15°, tan 75°

Sol. (*i*) $\sin 75° = \sin (45° + 30°)$

$= \sin 45° \cos 30° + \cos 45° \sin 30°$

$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}}$

(*ii*) $\cos 15° = \cos (45° - 30°) = \cos 45° \cos 30° + \sin 45° \sin 30°$

$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2}$

$= \frac{\sqrt{3}+1}{2\sqrt{2}}$

(*iii*) $\tan 75° = \tan (45° + 30°) = \frac{\tan 45° + \tan 30°}{1 - \tan 45° \tan 30°}$

$= \frac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}} = \frac{\sqrt{3}+1}{\sqrt{3}-1}$

Ex. 2 Show that $\frac{\sqrt{3} \cos 23° - \sin 23°}{2} = \cos 53°$

Sol. L. H. S. $= \frac{\sqrt{3} \cos 23° - \sin 23°}{2}$

$= \frac{\sqrt{3}}{2} \cos 23° - \frac{1}{2} \sin 23°$

But $\frac{\sqrt{3}}{2} = \cos 30°$ and $\frac{1}{2} = \sin 30°$

$\therefore$ L. H. S. $= \cos 30° \cos 23° - \sin 30° \sin 23°$

$= \cos (30° + 23°) = \cos 53° =$ R. H. S.

Ex. 3. Show that $\frac{\cos 9° - \sin 9°}{\cos 9° + \sin 9°} = \cot 54°$

Sol. $\dfrac{\text{Cos } 9^\circ - \text{Sin } 9^\circ}{\text{Cos } 9^\circ + \text{Sin } 9^\circ} = \dfrac{1 - \dfrac{\text{Sin } 9^\circ}{\text{Cos } 9^\circ}}{1 + \dfrac{\text{Sin } 9^\circ}{\text{Cos } 9^\circ}}$

[Divide num. and den. by Cos 9°]

$$= \frac{1 - \tan 9^\circ}{1 + \tan 9^\circ} = \frac{\tan 45^\circ - \tan 9^\circ}{1 + \tan 45^\circ \tan 9^\circ}$$

$(\because \tan 45^\circ = 1)$

$$= \tan (45^\circ - 9^\circ) = \tan 36^\circ$$

$$= \tan (90^\circ - 54^\circ) = \text{Cot } 54^\circ$$

Ex. 4. Prove that :—

$$\frac{\text{Sin (A—B)}}{\text{Sin A Sin B}} + \frac{\text{Sin (B—C)}}{\text{Sin B Sin C}} + \frac{\text{Sin (C—A)}}{\text{Sin C Sin A}} = 0$$

Sol. L. H. S. $= \dfrac{\text{Sin A Cos B—Cos A Sin B}}{\text{Sin A Sin B}}$

$$+ \frac{\text{Sin B Cos C—Cos B Sin C}}{\text{Sin B Sin C}} + \frac{\text{Sin C Cos A—Cos C Sin A}}{\text{Sin C Sin A}}$$

$$= \frac{\text{Sin A Cos B}}{\text{Sin A Sin B}} - \frac{\text{Cos A Sin B}}{\text{Sin A Sin B}} + \frac{\text{Sin B Cos C}}{\text{Sin B Sin C}}$$

$$- \frac{\text{Cos B Sin C}}{\text{Sin B Sin C}} + \frac{\text{Sin C Cos A}}{\text{Sin C Sin A}} - \frac{\text{Cos C Sin A}}{\text{Sin C Sin A}}$$

$=$ Cot B $-$ Cot A $+$ Cot C $-$ Cot B $+$ Cot A $-$ Cot C $= 0 =$ R. H. S.

Ex. 5. Show that :—

(*i*) $a \text{ Cos } \theta + b \text{ Sin } \theta = \sqrt{a^2 + b^2} \text{ Cos } (\theta - \phi)$ where $\tan \phi = \dfrac{b}{a}$

(*ii*) $\tan \alpha + \text{Cot } 2\alpha = \text{Cosec } 2\alpha$

(*iii*) $(1 + \tan A)(1 + \tan B) = 2$ if $A + B = 45^\circ$

Sol. (*i*) $a \text{ Cos } \theta+b \text{ Sin } \theta$

Put $a=r \text{ Cos } \varphi$......(*i*) and $b=r \text{ Sin } \varphi$......(*ii*)

Squaring and adding (*i*) and (*ii*) we get,

$$a^2+b^2=r^2(\text{Cos}^2\varphi \times \text{Sin}^2 \varphi) \quad \therefore r=\sqrt{a^2+b^2}$$

Also divide (*ii*) by (*i*), then we get

$$\tan \varphi=\frac{b}{a}$$

$$\therefore a \text{ Cos } \theta+b \text{ Sin } \theta=r [\text{Cos } \varphi \text{ Cos } \theta+\text{Sin } \varphi \text{ Sin } \theta]$$
$$=r \text{ Cos } (\theta-\varphi)$$
$$=\sqrt{a^2+b^2} \text{ Cos } (\theta-\varphi)$$

$$\text{where } \tan \varphi=\frac{b}{a}$$

(*ii*) $$\tan \alpha+\text{Cot } 2\alpha=\frac{\text{Sin } \alpha}{\text{Cos } \alpha}+\frac{\text{Cos } 2\alpha}{\text{Sin } 2\alpha}$$

$$=\frac{\text{Sin } 2\alpha \text{ Sin } \alpha+\text{Cos } 2\alpha \text{ Cos } \alpha}{\text{Sin } 2\alpha \text{ Cos } \alpha}$$

$$=\frac{\text{Cos } 2\alpha \text{ Cos } \alpha+\text{Sin } 2\alpha \text{ Sin } \alpha}{\text{Sin } 2\alpha \text{ Cos } \alpha}$$

$$=\frac{\text{Cos } (2\alpha-\alpha)}{\text{Sin } 2\alpha \text{ Cos } \alpha}=\frac{\text{Cos } \alpha}{\text{Sin } 2\alpha \text{ Cos } \alpha}$$

$$=\frac{1}{\text{Sin } 2\alpha}=\text{Cosec } 2\alpha$$

(*iii*) L.H.S.$=(1+\tan A) [1+\tan (45^\circ-A)]$

$$(\because A+B=45^\circ)$$

$$=(1+\tan A)\left[1+\frac{\tan 45^\circ-\tan A}{1+\tan 45^\circ \tan A}\right]$$

$$=(1+\tan A)\left[1+\frac{1-\tan A}{1+\tan A}\right]$$

$$(\because \tan 45^\circ=1)$$

$$=(1+\tan A)\left[\frac{1+\tan A+1-\tan A}{1+\tan A}\right]$$

$$=(1+\tan A)\frac{2}{(1+\tan A)}=2=\text{R.H.S.}$$

Ex. 6. If $\tan B=\dfrac{n \operatorname{Sin} A \operatorname{Cos} A}{1-n \operatorname{Sin}^2 A}$

prove that $\tan(A-B)=(1-n)\tan A$

Sol. We have $\tan B=\dfrac{n \operatorname{Sin} A \operatorname{Cos} A}{1-n \operatorname{Sin}^2 A}$

$$=\frac{n\dfrac{\operatorname{Sin} A \operatorname{Cos} A}{\operatorname{Cos}^2 A}}{\dfrac{1}{\operatorname{Cos}^2 A}-n\dfrac{\operatorname{Sin}^2 A}{\operatorname{Cos}^2 A}}$$

[Divide num. and denom. by $\operatorname{Cos}^2 A$]

$$=\frac{n\tan A}{\operatorname{Sec}^2 A-n\tan^2 A}=\frac{n\tan A}{1+(1-n)\tan^2 A}\quad\ldots(i)$$

[$\because 1+\tan^2 A=\operatorname{Sec}^2 A$]

Now L.H.S. $=\tan(A-B)=\dfrac{\tan A-\tan B}{1+\tan A\tan B}$

$$=\frac{\tan A-\dfrac{n\tan A}{1+(1-n)\tan^2 A}}{1+\tan A.\dfrac{n\tan A}{1+(1-n)\tan^2 A}}$$

[Using the result No : (i) for $\tan B$]

$$=\frac{\tan A+(1-n)\tan^3 A-n\tan A}{1+(1-n)\tan^2 A+n\tan^2 A}$$

$$=\frac{\tan A\,[1+(1-n)\tan^2 A-n]}{1+\tan^2 A}$$

$$=\frac{\tan A\,(1+\tan^2 A)\,(1-n)}{1+\tan^2 A}$$

$$=(1-n)\tan A=\text{R.H.S.}$$

Ex. 7. Prove that :

(*i*) $\tan 75° - \tan 30° - \tan 30° \tan 75° = 1$

(P. U. 1951)

(*ii*) $\tan 7A - \tan 4A - \tan 3A = \tan 3A . \tan 4A . \tan 7A$

(P.U. 1956)

Sol. (*i*) We have $45° = 75° - 30°$

$\therefore \tan 45° = \tan (75° - 30°)$

or $$1 = \frac{\tan 75° - \tan 30°}{1 + \tan 75° \tan 30°}$$

$(\because \tan 45° = 1)$

or $1 + \tan 75° \tan 30° = \tan 75° - \tan 30°$

$\therefore \tan 75° - \tan 30° - \tan 75° \tan 30° = 1.$

(*ii*) $7A = 4A + 3A$

$$\therefore \tan 7A = \tan (4A + 3A) = \frac{\tan 4A + \tan 3A}{1 - \tan 4A . \tan 3A}$$

which gives $\tan 7A - \tan 4A - \tan 3A = \tan 3A . \tan 4A . \tan 7A$.

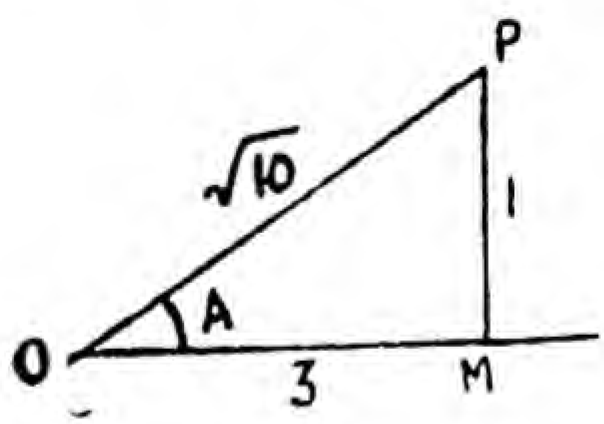
Ex. 8 If $\sin A = \dfrac{1}{\sqrt{10}}$ and

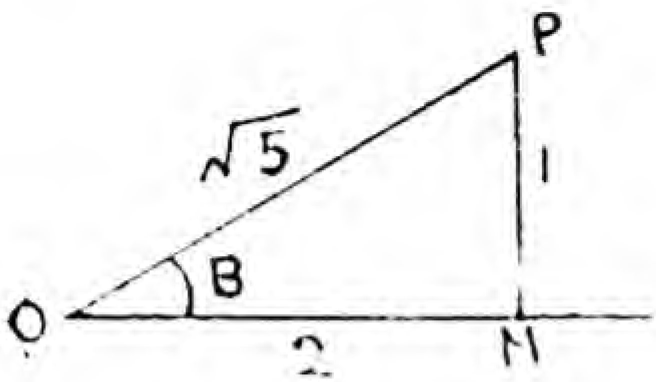
$\text{Sin } B = \dfrac{1}{\sqrt{5}}$

Prove that $A + B = 45°$ when A and B are in the *first* quadrant.

Sol. $\text{Sin } A = \dfrac{1}{\sqrt{10}}$

$\therefore \text{Cos } A = \dfrac{3}{\sqrt{10}}$





Again, $\text{Sin B} = \dfrac{1}{\sqrt{5}}$

$\therefore \quad \text{Cos B} = \dfrac{2}{\sqrt{5}}$

If $A+B=45^\circ$, then $\text{Sin}\,(A+B) = \dfrac{1}{\sqrt{2}}$

Now $\text{Sin}\,(A+B) = \text{Sin A Cos B} + \text{Cos A Sin B}$

$$= \frac{1}{\sqrt{10}} \cdot \frac{2}{\sqrt{5}} + \frac{3}{\sqrt{10}} \cdot \frac{1}{\sqrt{5}}$$

$$= \frac{2+3}{\sqrt{50}} = \frac{5}{5\sqrt{2}} = \frac{1}{\sqrt{2}}$$

which is true.

Hence $A+B=45^\circ$

EXERCISE III

1. Prove that (i) $\text{Cos}\,(A+45^\circ) = \text{Sin}\,(45^\circ - A)$

(ii) $\tan\left(\dfrac{\pi}{4} + \theta\right) = \dfrac{1+\tan\theta}{1-\tan\theta}$

(iii) $\sin\,(30^\circ - A) = \frac{1}{2}\,\text{Cos A} - \dfrac{\sqrt{3}}{2}\,\text{Sin A}$

2. Show that :—

(i) $\dfrac{\text{Cos}\,(\alpha+\beta)}{\text{Cos}\,\alpha\,\text{Cos}\,\beta} = 1 - \tan\alpha\tan\beta$

(ii) $\dfrac{\text{Sin}\,(\alpha-\beta)}{\text{Sin}\,\alpha\,\text{Sin}\,\beta} = \text{Cot}\,\beta - \text{Cot}\,\alpha$

(iii) $\dfrac{\text{Sin}\,(A+B)}{\text{Sin}\,(A-B)} = \dfrac{\tan A + \tan B}{\tan A - \tan B}$

(iv) $\tan 2\alpha - \tan\alpha = \tan\alpha\,\text{Sec}\,2\alpha$

(v) $\dfrac{\text{Sin}(\alpha-\beta)}{\text{Sin}\,\alpha\,\text{Sin}\,\beta}+\dfrac{\text{Sin}(\beta-\gamma)}{\text{Sin}\,\beta\,\text{Sin}\,\gamma}+\dfrac{\text{Sin}(\gamma-\alpha)}{\text{Sin}\,\gamma\,\text{Sin}\,\alpha}=0$

3. If $\text{Sin A}=\dfrac{3}{5}$ and $\text{Sin B}=\dfrac{4}{5}$, find $\text{Cos}(A-B)$ and $\text{Sin}(A-B)$, A and B being acute.

4. Prove that $\tan\alpha-\tan\beta=\dfrac{\sec\alpha\sec\beta}{\text{cosec}(\alpha-\beta)}$

5. The cosines of two angles of a triangle are $\frac{4}{5}$ and $\frac{12}{13}$ respectively. Find the cosine of the third angle.

6. The sines of two angles are $\frac{3}{5}$ and $\frac{5}{13}$, find the cosine of the 3rd angle.

7. Simplify into single terms :—

(i) $\cos(A+B)\cos A+\sin(A+B)\sin B$

(ii) $\sin(x+y)\cos x-\cos(x+y)\sin x$

(iii) $\sin 22^\circ\cos 38^\circ+\cos 22^\circ\sin 38^\circ$

(iv) $\sin(\theta+60^\circ)-\sin(\theta-60^\circ)$

8. Show that :—

(i) $\cos A+\cos(120^\circ-A)+\cos(120^\circ+A)=0$

(ii) $\dfrac{\tan\left(\frac{\pi}{4}+A\right)-\tan\left(\frac{\pi}{4}-A\right)}{\tan\left(\frac{\pi}{4}+A\right)+\tan\left(\frac{\pi}{4}-A\right)}=\sin 2A$

(iii) $\cos(60^\circ+\alpha)+\sin(30^\circ+\alpha)=\cos\alpha$

(iv) $\sin\alpha+\sin\left(\alpha+\frac{2\pi}{3}\right)+\sin\left(\alpha+\frac{4\pi}{3}\right)=0$

(v) $\cos(A+B)\cos B+\sin(A+B)\sin B=\cos A$

9. If $A+B=45^\circ$, show that

$(\cot A-1)(\text{Cot}\,B-1)=2$

(P.U. 1936)

10. Show that :—

(i) $\dfrac{\text{Cos } 11° + \text{Sin } 11°}{\text{Cos } 11° - \text{Sin } 11°} = \tan 56°$

(ii) $\dfrac{\text{Cos } 17° + \text{Sin } 17°}{\text{Cos } 17° - \text{Sin } 17°} = \tan 62°$

(iii) $\dfrac{\text{Cos } 37° + \text{Sin } 37}{\text{Cos } 37° - \text{Sin } 37°} = \text{Cot } 8°$ (*P.U. 1946*)

11. In a triangle ABC, if Sin C $=2$ Sin A Cos B, show that it is an isosceles triangle.

[Hint :—Sin C $=$ Sin $(A+B) =$ Sin A Cos B $+$ Cos A Sin B $= 2$ Sin A Cos B ...(Given)

$\because$ Sin A Cos B $-$ Cos A Sin B $=0$

or Sin $(A-B)=0$ which gives $A=B$]

12. If $A+B+C=\pi$, prove that

(i) $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

(ii) $\text{Cot } B \text{ Cot } C + \text{Cot } C \text{ Cot } A + \text{Cot } A \text{ Cot } B = 1$

(iii) $\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$

(*P.U. 1947*)

(iv) $\text{Cot }\frac{A}{2} + \text{Cot }\frac{B}{2} + \text{Cot }\frac{C}{2} = \text{Cot }\frac{A}{2} \text{ Cot }\frac{B}{2} \times \text{Cot }\frac{C}{2}$

13. Prove that :—

(i) $\tan 2A \tan 3A \tan 5A = \tan 5A - \tan 3A - \tan 2A$

(ii) $\tan\frac{\pi}{6} + \tan\frac{\pi}{12} + \tan\frac{\pi}{6} \tan\frac{\pi}{12} = 1$

(iii) $\tan 15° + \text{Cot } 15° = 4$

14. (i) If $\tan \alpha = x+1$ and $\tan \beta = x-1$ prove that $2 \text{ Cot } (\alpha-\beta) = x^2$

(*ii*) If 2 Cos $(\alpha+\beta)=\text{Sec }(\alpha-\beta)$, prove that

$$\text{Cot}^2 \beta=\frac{1+3\tan^2\alpha}{1-\tan^2\alpha}$$

15. (*i*) Prove that $\tan\theta \text{ Cot }\frac{\theta}{2}=1+\text{Sec }\theta$

(Hint :—Show that $\tan\theta \text{ Cot }\frac{\theta}{2}-1=\text{Sec }\theta$)

(*ii*) If $2\tan\beta+\text{Cot }\beta=\tan\alpha$ prove that $\text{Cot }\beta=2\tan(\alpha-\beta)$

(*Allahabad 1935*)

CHAPTER IV

Trigonometrical Ratios of Multiple and Sub-multiple Angles.

4.1. Multiple angles.

4.11. To prove that :—

(*i*) $Sin\ 2A = 2\ Sin\ A\ Cos\ A$

(*ii*) $Cos\ 2A = Cos^2 A - Sin^2 A$

$= 2\ Cos^2 A - 1$

$= 1 - 2\ Sin^2 A$

(*iii*) $tan\ 2A = \dfrac{2\ tan\ A}{1 - tan^2 A}$

Proof : (*i*) We have $Sin\ (A+B) = Sin\ A\ Cos\ B + Cos\ A\ Sin\ B$.

Putting $A = B$, we get

$Sin\ (A+A) = Sin\ A\ Cos\ A + Cos\ A\ Sin\ A$

$\therefore\quad Sin\ 2A = 2\ Sin\ A\ Cos\ A$

Cor. $Sin\ A = 2\ Sin \dfrac{A}{2}\ Cos\ \dfrac{A}{2}$

Hint . $Sin\ A = Sin\ \left(\dfrac{A}{2} + \dfrac{A}{2}\right)$ and proceed as in (*i*)

(*ii*) $Cos\ (A+B) = Cos\ A\ Cos\ B - Sin\ A\ Sin\ B$

Putting $A = B$, we have

$Cos\ (A+A) = Cos\ A\ Cos\ A - Sin\ A\ Sin\ A$

$\therefore\quad Cos\ 2A = Cos^2 A - Sin^2 A \qquad \ldots\ldots (A)$

$= Cos^2 A - (1 - Cos^2 A)$

$$= 2\,Cos^2 A - 1 \qquad \ldots\ldots(B)$$

$$= 2(1 - Sin^2 A) - 1$$

$$= 1 - 2\,Sin^2 A \qquad \ldots\ldots(C)$$

Cor. $\quad \text{Cos } A = \text{Cos}^2 \frac{A}{2} - \text{Sin}^2 \frac{A}{2}$

$$= 2\,\text{Cos}^2 \frac{A}{2} - 1$$

$$= 1 - 2\,\text{Sin}^2 \frac{A}{2}$$

Hint : $\quad \text{Cos } A = \text{Cos}\left(\frac{A}{2} + \frac{A}{2}\right)$ and

proceed as in (*ii*)

(*iii*) We have $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

Putting $A = B$, we have

$$\tan(A+A) = \frac{\tan A + \tan A}{1 - \tan A \tan A}$$

$$\therefore \quad \tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

4.1.2. To prove that

(*i*) $\text{Sin } 2A = \dfrac{2 \tan A}{1 + \tan^2 A}$

(*ii*) $\text{Cos } 2A = \dfrac{1 - \tan^2 A}{1 + \tan^2 A}$

Proof : (*i*) We have seen that

$$\text{Sin } 2A = 2 \text{ Sin A Cos A}$$

Now $2 \text{ Sin A Cos A} = \dfrac{2 \text{ Sin A Cos A}}{\text{Cos}^2 A + \text{Sin}^2 A}$

$(\because \text{Cos}^2 A + \text{Sin}^2 A = 1$

$$= \frac{2\,\frac{\text{Sin A Cos A}}{\text{Cos}^2\text{ A}}}{\frac{\text{Cos}^2\text{ A}}{\text{Cos}^2\text{ A}} + \frac{\text{Sin}^2\text{ A}}{\text{Cos}^2\text{ A}}}$$

$$= \frac{2\tan A}{1+\tan^2 A}$$

(ii) We have proved that

$$\text{Cos } 2A = \text{Cos}^2 A - \text{Sin}^2 A$$

Now $\text{Cos}^2 A - \text{Sin}^2 A = \dfrac{\text{Cos}^2 A - \text{Sin}^2 A}{\text{Cos}^2 A + \text{Sin}^2 A}$

$(\because \text{Cos}^2 A + \text{Sin}^2 A = 1)$

$$= \frac{\frac{\text{Cos}^2 A}{\text{Cos}^2 A} - \frac{\text{Sin}^2 A}{\text{Cos}^2 A}}{\frac{\text{Cos}^2 A}{\text{Cos}^2 A} + \frac{\text{Sin}^2 A}{\text{Cos}^2 A}}$$

$$= \frac{1-\tan^2 A}{1+\tan^2 A}$$

4.1.3. Please note that :—

(i) $\text{Sin A} = \dfrac{2\tan\frac{A}{2}}{1+\tan^2\frac{A}{2}}$

(ii) $\text{Cos A} = \dfrac{1-\tan^2\frac{A}{2}}{1+\tan^2\frac{A}{2}}$

and (iii) $\tan A = \dfrac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}}$

4.2 Prove **geometrically** that

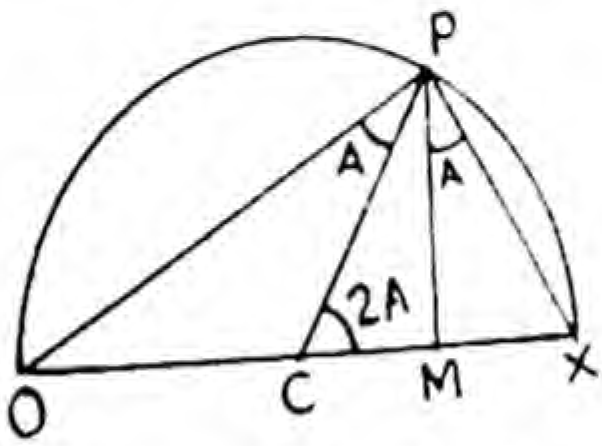
(*i*) $\text{Sin } 2A = 2 \text{ Sin } A \text{ Cos } A$

(*ii*) $\text{Cos } 2A = \text{Cos}^2 A - \text{Sin}^2 A$

$= 2 \text{ Cos}^2 A - 1$

$= 1 - 2 \text{ Sin}^2 A$

(*iii*) $\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}$



Take $\angle XOP = A$. With any point C on OX as centre and radius equal to CO $(= a)$ draw a circle cutting OP and OX in P and X respectively. Join OP, PX and draw PM ⊥ on OX.

Then $\angle CPO = \angle COP = A$ $(\because OC = CP)$

$\angle XCP = \angle CPO + \angle COP = 2A$..................(Why ?)

and $\angle XPM = 90° - \angle PXM = \angle COP = A$

(*i*) Now $\dfrac{MP}{PC} = \text{Sin } \angle PCM = \text{Sin } 2A$

$\therefore MP = PC \text{ Sin } 2A = a \text{ Sin } 2A$

Also $MP = OX. \dfrac{OP}{OX} \cdot \dfrac{MP}{OP} = 2a \text{ Cos } A \text{ Sin } A$

(Please note this step)

$\therefore a \text{ Sin } 2A = 2a \text{ Sin } A \text{ Cos } A$

i. e. $\text{Sin } 2A = 2 \text{ Sin } A \text{ Cos } A$

(*ii*) Again, $OM = OC + CM$

and $MX = CX - CM$

$\therefore$ $OM - MX = 2CM$.........(1) (by subtraction)

Also $OM = OX. \dfrac{OP}{OX} \cdot \dfrac{OM}{OP} = 2a \text{ Cos } A \text{ Cos } A$

(Please note this step)

$= 2a \text{ Cos}^2\text{A}$.........(2)

and $\quad \text{MX} = \text{OX}.\ \dfrac{\text{PX}}{\text{OX}} \cdot \dfrac{\text{MX}}{\text{PX}} = 2a \text{ Sin A Sin A}$

(Please note this step)

$= 2a \text{ Sin}^2 \text{A}$.........(3)

Also $\quad \text{CM} = \text{CP Cos 2A} = a \text{ Cos 2A}$.........(4)

Subtracting (3) from (2) we get

$\text{OM} - \text{MX} = 2a\ (\text{Cos}^2\text{A} - \text{Sin}^2\text{A})$ ....(5)

But $\text{OM} - \text{MX} = 2\text{CM} = 2a \text{ Cos 2A}$ ....from (4)

$\therefore 2a \text{ Cos 2A} = 2a\ (\text{Cos}^2\text{A} - \text{Sin}^2\text{A})$

or $Cos\ 2A = Cos^2 A - Sin^2 A$

Again $\quad \text{CM} = \text{OM} - \text{OC} = \text{OM} - a$ ....(6)

and also $\quad \text{CM} = \text{CX} - \text{MX} = a - \text{MX}$ ....(7)

But from (4) $\qquad \text{CM} = a \text{ Cos 2A}$

and from (2) $\qquad \text{OM} = 2a \text{ Cos}^2\text{A}$

Substituting these values of CM and OM in (6) we get

$$a \text{ Cos}^2\text{A} = 2a \text{ Cos}^2\text{A} - a$$

$$\therefore Cos^2 A = 2Cos^2 A - 1$$

Also from (3) $\qquad \text{MX} = 2a \text{ Sin}^2 \text{A}$

$\therefore$ Substituting the values of MX and CM in (7) we get

$$a \text{ Cos2A} = a - 2a \text{ Sin}^2\text{A}$$

$$\therefore Cos2A = 1 - 2\ Sin^2 \text{A}$$

(iii) $\quad \tan 2\text{A} = \dfrac{\text{MP}}{\text{CM}} = \dfrac{2\text{MP}}{2\text{CM}} = \dfrac{2\text{MP}}{\text{OM} - \text{MX}}$

[using the result (1)]

$$= \frac{2\dfrac{\text{MP}}{\text{OM}}}{1 - \dfrac{\text{MX}}{\text{MP}} \cdot \dfrac{\text{MP}}{\text{OM}}} = \frac{2\tan \text{A}}{1 - \tan^2 \text{A}}$$

Note :—*These results can also conveniently be obtained by making $B=A$ in Article 3.1 chapter III.*

4.2.1. Two Important Results.

$$1+\cos 2A=2\cos^2 A,$$

$$1-\cos 2A=2\sin^2 A$$

Note :—The student is advised to commit these results to memory as he has to use these frequently in solving a number of questions.

4.3. To prove that

(*i*) $\sin(A+B)\sin(A-B)=\sin^2 A-\sin^2 B$ ......(1)

$=\cos^2 B-\cos^2 A$ ......(2)

(*ii*) $\cos(A+B)\cos(A-B)=\cos^2 A-\sin^2 B$ ......(3)

$=\cos^2 B-\sin^2 A$ ......(4)

(*P. U. 1941, 1944*)

Proof :—

(*i*) $\sin(A+B)\sin(A-B)$

$=(\sin A\cos B+\cos A\sin B)(\sin A\cos B-\cos A\sin B)$

$=\sin^2 A\cos^2 B-\cos^2 A\sin^2 B$

$=\sin^2 A(1-\sin^2 B)-(1-\sin^2 A)\sin^2 B$ ......(1)

$=\sin^2 A-\sin^2 B$

$=(1-\cos^2 A)-(1-\cos^2 B)$ ......(2)

$=\cos^2 B-\cos^2 A$

(*ii*) $\cos(A+B)\cos(A-B)$

$=(\cos A\cos B-\sin A\sin B)(\cos A\cos B+\sin A\sin B)$

$=\cos^2 A\cos^2 B-\sin^2 A\sin^2 B$

$=\cos^2 A(1-\sin^2 B)-(1-\cos^2 A)\sin^2 B$ ......(3)

$=\cos^2 A-\sin^2 B$

$=(1-\sin^2 A)-(1-\cos^2 B)$ ......(4)

$=\cos^2 B-\sin^2 A$

Solved Examples

Ex. 1. Show that :—

(i) $\dfrac{\text{Sin } 2A}{1+\text{Cos } 2A}=\tan A$

(ii) $\dfrac{\text{Sin } 2A}{1-\text{Cos } 2A}=\text{Cot } A$

(iii) $\dfrac{1-\text{Cos } 2A}{1+\text{Cos } 2A}=\tan^2 A$

Sol. (i) L. H. S.

$$=\frac{\text{Sin } 2A}{1+\text{Cos } 2A}=\frac{2 \text{ Sin } A \text{ Cos } A}{1+(2 \text{ Cos}^2 A-1)}$$

$$=\frac{2 \text{Sin } A \text{ Cos } A}{2 \text{ Cos}^2 A}=\frac{\text{Sin } A}{\text{Cos } A}=\tan A=\text{R. H. S}$$

(ii) L. H. S $=\dfrac{\text{Sin } 2A}{1-\text{Cos } 2A}=\dfrac{2 \text{ Sin} A \text{ Cos } A}{1-(1-2 \text{ Sin}^2 A)}$

$$=\frac{2 \text{ Sin } A \text{ Cos } A}{1-1+2 \text{ Sin}^2 A}=\frac{2 \text{ Sin } A \text{ Cos } A}{2 \text{ Sin}^2 A}$$

$$=\frac{\text{Cos } A}{\text{Sin } A}=\text{Cot } A=\text{R. H. S.}$$

(iii) L. H. S. $=\dfrac{1-\text{Cos } 2A}{1+\text{Cos } 2A}=\dfrac{1-(1-2 \text{ Sin}^2 A)}{1+(2 \text{ Cos}^2 A-1)}$

$$=\frac{2 \text{ Sin}^2 A}{2 \text{ Cos}^2 A}=\tan^2 A=\text{R. H. S.}$$

Note :— *The student is advised to remember that :—*

(i) $1+\text{Cos } 2A=2 \text{ Cos}^2 A$

and (ii) $1-\text{Cos } 2A=2 \text{ Sin}^2 A$

Ex. 2. Prove that

$$\left(\text{Sin } \frac{A}{2}+\text{Cos } \frac{A}{2}\right)^2=1+\text{Sin } A$$

Sol. L. H. S. $=\left(\operatorname{Sin}\frac{A}{2}+\operatorname{Cos}\frac{A}{2}\right)^2$

$$=\operatorname{Sin}^2\frac{A}{2}+\operatorname{Cos}^2\frac{A}{2}+2\operatorname{Sin}\frac{A}{2}\operatorname{Cos}\frac{A}{2}$$

$$=1+2\operatorname{Sin}\frac{A}{2}\operatorname{Cos}\frac{A}{2}\left(\because \operatorname{Sin}^2\frac{A}{2}+\operatorname{Cos}^2\frac{A}{2}=1\right)$$

But $2\operatorname{Sin}\frac{A}{2}\operatorname{Cos}\frac{A}{2}=\operatorname{Sin} A$ ......(why ?

$\therefore$ L. H. S. $=1+\operatorname{Sin} A=$ R. H. S.

Ex. 3. Prove that :—

(*i*) $1-\operatorname{Sin} 2\theta=(\operatorname{Sin}\theta-\operatorname{Cos}\theta)^2$

and (*ii*) $\operatorname{Cos}^4\theta-\operatorname{Sin}^4\theta=\operatorname{Cos} 2\theta$

Sol. (*i*) L. H. S. $=1-\operatorname{Sin} 2\theta$

$$=\operatorname{Sin}^2\theta+\operatorname{Cos}^2\theta-2\operatorname{Sin}\theta\operatorname{Cos}\theta$$

$$(\because \operatorname{Sin}^2\theta+\operatorname{Cos}^2\theta=1)$$

$$=(\operatorname{Sin}\theta-\operatorname{Cos}\theta)^2=\text{R. H. S.}$$

(*ii*) L. H. S. $=\operatorname{Cos}^4\theta-\operatorname{Sin}^4\theta=(\operatorname{Cos}^2\theta+\operatorname{Sin}^2\theta)(\operatorname{Cos}^2\theta-\operatorname{Sin}^2\theta)$

$$=\operatorname{Cos}^2\theta-\operatorname{Sin}^2\theta \quad (\because \operatorname{Cos}^2\theta+\operatorname{Sin}^2\theta=1)$$

$$=\operatorname{Cos} 2\theta=\text{R. H. S.}$$

Ex. 4. Prove that :—

$\operatorname{Sin}(2n+1)\theta\operatorname{Sin}\theta=\operatorname{Sin}^2(n+1)\theta-\operatorname{Sin}^2 n\theta$

Sol. R. H. S. $=\operatorname{Sin}^2(n+1)\theta-\operatorname{Sin}^2 n\theta$

Put $(n+1)\theta=A$ and $n\theta=B$

$\therefore$ R. H. S. $=\operatorname{Sin}^2 A-\operatorname{Sin}^2 B$

$$=\operatorname{Sin}(A+B)\operatorname{Sin}(A-B)$$

Replacing A and B, we get

R. H. S. $=\operatorname{Sin}(\overline{n+1}\,\theta+n\theta)\operatorname{Sin}(\overline{n+1}\,\theta-n\theta)$

$$=\operatorname{Sin}(2n+1)\theta\operatorname{Sin}\theta=\text{L. H. S.}$$

Ex. 5. If $\tan\frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{u}{2}$, show that

$$\text{Cos}\,\theta = \frac{\text{Cos}\,u - e}{1 - e\,\text{Cos}\,u}$$

Sol. $\tan\frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{u}{2}$

Squaring, we get

$$\tan^2\frac{\theta}{2} = \frac{(1+e)\tan^2\frac{u}{2}}{1-e}$$

or $\dfrac{1}{\tan^2\frac{\theta}{2}} = \dfrac{(1-e)}{(1+e)\tan^2\frac{u}{2}}$ (By reversing the two sides)

$$\text{or}\quad \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} = \frac{(1-e)-(1+e)\tan^2\frac{u}{2}}{(1-e)+(1+e)\tan^2\frac{u}{2}}$$

(By divido and compo.)

$$\text{or}\quad \text{Cos}\,\theta = \frac{\left(1-\tan^2\frac{u}{2}\right)-e\left(1+\tan^2\frac{u}{2}\right)}{\left(1+\tan^2\frac{u}{2}\right)-e\left(1-\tan^2\frac{u}{2}\right)}$$

$$= \frac{\dfrac{1-\tan^2\frac{u}{2}}{1+\tan^2\frac{u}{2}}-e}{1-e\,\dfrac{1-\tan^2\frac{u}{2}}{1+\tan^2\frac{u}{2}}} = \frac{\text{Cos}\,u - e}{1 - e\,\text{Cos}\,u}$$

EXERCISE IV

1. If Cos A $=\frac{3}{5}$, find Cos 2 A
2. If Sin A $=\frac{1}{7}$, find Cos 2A
3. If Sin A $=\frac{12}{13}$, find Sin 2A
4. If tan $\theta=5$, find tan 2θ
5. If tan $\theta=2$, find Sin 2θ and Cos 2θ
6. Prove that :—

   (*i*) $\operatorname{Sec} 2A+\tan 2A=\tan\left(\frac{\pi}{4}+A\right)$

   (*ii*) $\dfrac{1+\operatorname{Sin} 2\theta}{1-\operatorname{Sin} 2\theta}=\tan^2\left(\frac{\pi}{4}+\theta\right)$

7. Prove that :—

   (*i*) $\dfrac{1+\operatorname{Sin} 2\theta-\operatorname{Cos} 2\theta}{1+\operatorname{Sin} 2\theta+\operatorname{Cos} 2\theta}=\tan\theta$

   (*ii*) $\dfrac{\operatorname{Sin}\theta+\operatorname{Sin} 2\theta}{1+\operatorname{Cos}\theta+\operatorname{Cos} 2\theta}=\tan\theta$

   (*iii*) $\dfrac{\operatorname{Cos}\theta+\operatorname{Sin} 2\theta}{1-\operatorname{Cos} 2\theta+\operatorname{Sin} 2\theta}=\operatorname{Cot}\theta$

   (*iv*) $\dfrac{1+\operatorname{Sin}\theta-\operatorname{Cos}\theta}{1+\operatorname{Sin}\theta+\operatorname{Cos}\theta}=\tan\frac{\theta}{2}$

8. If a Sin $\theta=b$ Cos θ, find the value of a Cos $2\theta+b$ Sin 2θ.
(*P.U. 1948*)

9. (*a*) If Cos $\theta=\frac{1}{2}\left(a+\frac{1}{a}\right)$

   Show that Cos $2\theta=\frac{1}{2}\left(a^2+\frac{1}{a^2}\right)$ (*P.U. 1946*)

   (*b*) Prove that $2\operatorname{Cos}\theta=\sqrt{2+\sqrt{(2+2\operatorname{Cos} 4\theta)}}$

10. Show that :—

    (*i*) $\tan 50^\circ=2\tan 10^\circ+\tan 40^\circ$

    (*ii*) $\tan 70^\circ=2\tan 50^\circ+\tan 20^\circ$

11. If $\frac{a}{b}$ = Sec 2A, prove that

$$\sqrt{\frac{a-b}{a+b}}+\sqrt{\frac{a+b}{a-b}}=\frac{2}{\text{Sin } 2A}$$

12. Prove that :—

(*i*) Sin (2A−B) Cos (2B−A) + Cos (2A−B) Sin (2 B−A) = Sin (A+B)

(*ii*) $\text{Cos}^2(45°-B)-\text{Sin}^2(45°-A)=\text{Sin }(A+B)\text{ Cos }(A-B)$

(*iii*) $$\frac{\text{Sin}^2 A-\text{Sin}^2 B}{\text{Cos}^2 A-\text{Sin}^2 B}=\frac{\tan (A+B)}{\text{Cot }(A-B)}$$

(*iv*) $$\frac{\text{Sin } 3\theta \text{ Cos } \theta-\text{Cos } 3\theta \text{ Sin } \theta}{\text{Cos}^2 2\theta-\text{Sin}^2 2\theta}=\tan 4\theta$$

(*v*) $$\frac{\text{Sin } 4\theta}{1+\text{Cos } 4\theta}-\frac{1-\text{Cos } 4\theta}{\text{Sin } 4\theta}\ \tan^2\theta$$

13. If Sin $\theta=\frac{a-b}{a+b}$, find $\tan\frac{\theta}{2}$ (*P.U. 1952*)

14. If $\tan\frac{\theta}{2}=\sqrt{\frac{1+e}{1-e}}\tan\frac{u}{2}$, show that

$$\text{Cos }\theta=\frac{\text{Cos } u-e}{1-e\text{ Cos } u}$$

Hint : See Ex. 5 (solved)

15. Prove that tan 15° + Cot 15° = 4.

[**Hint** : $\tan 15°+\text{Cot } 15°=\frac{\text{Sin } 15°}{\text{Cos } 15°}+\frac{\text{Cos } 15°}{\text{Sin } 15°}$ and proceed]

4.4 To prove that :—

(*i*) *Sin 3A = 3 Sin A − 4 Sin³ A*

(*ii*) *Cos 3 A = 4 Cos³ A − 3 Cos A*

and (*iii*) $\tan 3A=\frac{3\tan A-\tan^3 A}{1-3\tan^2 A}$

Proof. (*i*) $\sin 3A = \sin (2A + A) = \sin 2A \cos A + \cos 2A \sin A$

$= 2 \sin A \cos A . \cos A + (1 - 2 \sin^2 A) \sin A$

$= 2 \sin A \cos^2 A + (1 - 2 \sin^2 A) \sin A$

$= 2 \sin A (1 - \sin^2 A) + (1 - 2 \sin^2 A) \times \sin A$

$= 3 \sin A - 4 \sin^3 A$

(*ii*) $\cos 3A = \cos (2A + A) = \cos 2A \cos A - \sin 2A \sin A$

$= (2 \cos^2 A - 1) \cos A - 2 \sin A \cos A \times \sin A$

$= (2 \cos^2 A - 1) \cos A - 2 \sin^2 A \cos A$

$= (2 \cos^2 A - 1) \cos A - 2(1 - \cos^2 A) \cos A$

$= 4 \cos^3 A - 3 \cos A$

(*iii*) $\tan 3A = \tan (2A + A) = \dfrac{\tan 2A + \tan A}{1 - \tan 2A \tan A}$

$$= \frac{\dfrac{2 \tan A}{1 - \tan^2 A} + \tan A}{1 - \dfrac{2 \tan A}{1 - \tan^2 A} . \tan A}$$

$$= \frac{\dfrac{3 \tan A - \tan^3 A}{1 - \tan^2 A}}{\dfrac{1 - 3 \tan^2 A}{1 - \tan^2 A}}$$

$$= \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$$

Otherwise thus :—

(*iv*) $\tan 3A = \dfrac{\sin 3A}{\cos 3A} = \dfrac{3 \sin A - 4 \sin^3 A}{4 \cos^3 A - 3 \cos A}$

$$= \frac{3 \dfrac{\sin A}{\cos^3 A} - 4 \dfrac{\sin^3 A}{\cos^3 A}}{4 \dfrac{\cos^3 A}{\cos^3 A} - 3 \dfrac{\cos A}{\cos^3 A}} = \frac{3 \tan A \sec^2 A - 4 \tan^3 A}{4 - 3 \sec^2 A}$$

$$= \frac{3 \tan A (1+\tan^2 A) - 4 \tan^3 A}{4 - 3 (1+\tan^2 A)}$$

$$= \frac{3 \tan A - \tan^3 A}{1 - 3\tan^2 A}$$

Solved Examples

Ex. 1. If $2 \operatorname{Cos} \theta = a + \frac{1}{a}$, prove that

$$\operatorname{Cos} 3\theta = \frac{1}{2} \left(a^3 + \frac{1}{a^3} \right)$$

Sol. $2 \operatorname{Cos} \theta = a + \frac{1}{a}$

Cubing both sides, we have

$$8 \operatorname{Cos}^3\theta = a^3 + \frac{1}{a^3} + 3.a.\frac{1}{a}\left(a + \frac{1}{a} \right)$$

$$= a^3 + \frac{1}{a^3} + 3 \cdot 2 \operatorname{Cos} \theta$$

$$= a^3 + \frac{1}{a^3} + 6 \operatorname{Cos} \theta$$

or $8 \operatorname{Cos}^3\theta - 6 \operatorname{Cos} \theta = a^3 + \frac{1}{a^3}$

or $2(4 \operatorname{Cos}^3 \theta - 3 \operatorname{Cos} \theta) = a^3 + \frac{1}{a^3}$

or $4 \operatorname{Cos}^3 \theta - 3 \operatorname{Cos} \theta = \frac{1}{2} \left(a^3 + \frac{1}{a^3} \right)$

or $\qquad \operatorname{Cos} 3\theta = \frac{1}{2} \left(a^3 + \frac{1}{a^3} \right)$

Ex. 2. Find the value of tan 15° from the equation
$3 \tan \theta - 3 \tan^3 \theta = 1 - 3 \tan^2 \theta$

Sol. We know $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$

Put $\theta = 15^\circ$

$\therefore \tan 3\theta = \tan 45^\circ = 1 = \dfrac{3\tan 15^\circ - \tan^3 15^\circ}{1-3\tan^2 15^\circ}$

$\therefore 1-3\tan^2 15^\circ = 3\tan 15^\circ - \tan^3 15^\circ$

or $\tan^3 15^\circ - 3\tan^2 15^\circ - 3\tan 15^\circ + 1 = 0$

or $(\tan^3 15^\circ + 1) - (3\tan^2 15^\circ + 3\tan 15^\circ) = 0$

or $(\tan 15^\circ + 1)(\tan^2 15^\circ - \tan 15^\circ + 1) - 3\tan 15^\circ (\tan 15^\circ + 1) = 0$

or $(\tan 15^\circ + 1)(\tan^2 15^\circ - 4\tan 15^\circ + 1) = 0$

Now $\tan 15^\circ + 1 \neq 0$

$\therefore \tan 15^\circ \neq -1$

$\therefore \tan^2 15^\circ - 4\tan 15^\circ + 1 = 0$

or $\tan 15^\circ = \dfrac{4 \pm \sqrt{16-4}}{2}$

$= \dfrac{4 \pm 2\sqrt{3}}{2}$

$= 2 \pm \sqrt{3}$

Now $\tan 15^\circ < \tan 45^\circ$ *i. e.*, $\tan 15^\circ < 1$

$\therefore \tan 15^\circ = 2 - \sqrt{3}$

EXERCISE V

Show that :—

1. $\sin A \sin(60^\circ - A)\sin(60^\circ + A) = \frac{1}{4}\sin 3A$
2. $\cos A \cos(60^\circ - A)\cos(60^\circ + A) = \frac{1}{4}\cos 3A$
3. $\dfrac{1}{\tan\theta + \tan 3\theta} - \dfrac{1}{\cot\theta + \cot 3\theta} = \cot 4\theta$
4. Prove that $(3\sin A - \sin 3A)^{\frac{2}{3}} + (3\cos A + \cos 3A)^{\frac{2}{3}} = 4^{\frac{2}{3}}$

5. $4 \sin A \sin\left(A-\frac{\pi}{3}\right) \sin\left(A-\frac{2\pi}{3}\right)=\sin 3A$

6. $\sin^3 \theta+\sin^3 (120^\circ+\theta)+\sin^3 (240^\circ+\theta)=-\frac{3}{4} \sin 3\theta$

7. $\tan A+\tan (60^\circ+A)+\tan (120^\circ+A)=3 \tan 3A$

8. If $x^2+y^2=1$, show that

$$(3x-4x^3)^2+(3y-4y^3)^2=1$$

[**Hint** :—Put $x=\sin \theta$ and $y=\cos \theta$]

4.5. Sub Multiple Angles.

4.5.1. *To Express trigonometrical functions of an angle in terms of the* **cosine** *of double the angle.*

We have $\cos 2A=2 \cos^2 A-1$

$$\text{or } \cos^2 A=\frac{1+\cos 2A}{2}$$

$$\therefore \cos A=\pm\sqrt{\frac{1+\cos 2A}{2}} \qquad \ldots\ldots(1)$$

Again, $\cos 2A=1-2 \sin^2 A$

$$\text{or } 2 \sin^2 A=1-\cos 2A$$

$$\therefore \sin A=\pm\sqrt{\frac{1-\cos 2A}{2}} \qquad \ldots\ldots(2)$$

Dividing (2) by (1) we have

$$\tan A=\pm\sqrt{\frac{1-\cos 2A}{1+\cos 2A}} \qquad \ldots\ldots(3)$$

Similarly, we can find Sec A, Cosec A, and Cot A by taking the reciprocals of (1), (2) and (3) respectively.

Important Note. *The ambiguity of signs in the above results cannot be removed, unless the quadrant in which A lies is known to us.*

For instance, $\cos A=\sqrt{\frac{1+\cos 2A}{2}}$, *if A lies either in the 1st*

or in the fourth quadrant. But Cos $A=-\sqrt{\frac{1+\cos 2A}{2}}$ *if* A *lies in the second quadrant or in the third quadrant and so on.*

This can be illustrated by means of the following example.

Ex. Find the values of Sin 22° 30' and tan 22° 30'

Sol. Put A=22° 30' $\therefore 2A=45°$

$$\therefore \sin 22° \ 30'=\sqrt{\frac{1-\cos 45°}{2}}$$

$$=\sqrt{\frac{1-\frac{1}{\sqrt{2}}}{2}}=\frac{1}{2}\sqrt{2-\sqrt{2}} \quad \ldots\ldots\ldots(i)$$

$$\text{Again } \cos 22° \ 30'=\sqrt{\frac{1+\cos 45°}{2}}=\sqrt{\frac{1+\frac{1}{\sqrt{2}}}{2}}$$

$$=\tfrac{1}{2}\sqrt{2+\sqrt{2}} \quad \ldots(2)$$

$\therefore$ from (1) and (2), we have

$$\tan 22° \ 30' = \frac{\sqrt{2-\sqrt{2}}}{\sqrt{2+\sqrt{2}}}$$

Note :—We have taken the sign before the radicals because angle A is acute.

4.5.2. *To express the trigonometrical functions of an angle in terms of the* **Sine** *of double the angle.*

We know $\sin^2 A+\cos^2 A=1$ ...(1)

and $2 \sin A \cos A=\sin 2A$ ...(2)

$\therefore$ Adding (1) and (2) we get

$\sin^2 A+\cos^2 A+2 \sin A \cos A=1+\sin 2A$

or $(\sin A+\cos A)^2=1+\sin 2A$

$\therefore \sin A+\cos A=\pm\sqrt{1+\sin 2A}$......(3)

Similarly, subtracting (2) from (1) and taking the square root, we get.

$$\text{Sin A} - \text{Cos A} = \pm\sqrt{1 - \text{Sin 2A}} \ldots\ldots (4)$$

From (3) and (4) adding and Subtracting, we get

$$\text{Sin A} = \tfrac{1}{2}\left[\pm\sqrt{1+\text{Sin 2A}} \pm \sqrt{1-\text{Sin 2A}}\right] \quad \ldots (5)$$

$$\text{and Cos A} = \tfrac{1}{2}\left[\pm\sqrt{1+\text{Sin 2A}} \mp \sqrt{1-\text{Sin 2A}}\right] \quad \ldots (6)$$

Dividing (5) by (6) we get.

$$\tan A = \frac{(\pm\sqrt{1+\text{Sin 2A}} \pm \sqrt{1-\text{Sin 2A}})}{(\pm\sqrt{1+\text{Sin 2A}} \mp \sqrt{1-\text{Sin 2A}})} \quad \ldots (7)$$

Taking reciprocals of (5), (6) and (7) we can get Cosec A, Sec A and Cot A respectively.

4.5.3. *To express the trigonometrical functions of an angle in terms of the* **tangent** *of double the angle.*

Here we have $\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}$

or $\quad \tan 2A \tan^2 A + 2 \tan A - \tan 2A = 0$

$$\therefore \quad \tan A = \frac{-2 \pm \sqrt{4 + 4 \tan^2 2A}}{2 \tan 2A}$$

$$= \frac{-1 \pm \sqrt{1 + \tan^2 2A}}{\tan 2A}$$

With the help of tan A having thus been found, we can easily find other trigonometrical functions.

4.5.4. From the results 4.5.1 – 4.5.3 (or quite independently), we can easily prove that :—

(a) (i) $\text{Cos}\dfrac{A}{2} = \pm\sqrt{\dfrac{1+\text{Cos A}}{2}}$

(ii) $\text{Sin}\dfrac{A}{2} = \pm\sqrt{\dfrac{1-\text{Cos A}}{2}}$,

(iii) $\tan\dfrac{A}{2} = \pm\sqrt{\dfrac{1-\text{Cos A}}{1+\text{Cos A}}}$, and so on.

(b) (i) $\text{Sin}\,\frac{A}{2}=\frac{1}{2}\left\{\pm\sqrt{1+\text{Sin A}}\pm\sqrt{1-\text{Sin A}}\right\}$

(ii) $\text{Cos}\,\frac{A}{2}=\frac{1}{2}\left\{\pm\sqrt{1+\text{Sin A}}\mp\sqrt{1-\text{Sin A}}\right\}$

(iii) $\tan\frac{A}{2}=\frac{(\pm\sqrt{1+\text{Sin A}}\pm\sqrt{1-\text{Sin A}})}{(\pm\sqrt{1+\text{Sin A}}\mp\sqrt{1-\text{Sin A}})}$, and so on

and (c) $\tan\frac{A}{2}=\frac{-1\pm\sqrt{1+\tan^2 A}}{\tan A}$

These are left as an exercise for the student.

4.6.1. Trigonometric functions of 18° and 72°

Let $18°=\theta$, so that $5\theta=90°$

Now $2\theta=90°-3\theta$

$\therefore$ $\text{Sin}\,2\theta=\text{Sin}\,(90°-3\theta)=\text{Cos}\,3\theta=4\,\text{Cos}^3\theta-3\,\text{Cos}\,\theta$

or $2\,\text{Sin}\,\theta\,\text{Cos}\,\theta=4\,\text{Cos}^3\theta-3\,\text{Cos}\,\theta$

or $2\,\text{Sin}\,\theta=4\,\text{Cos}^2\theta-3$ (Dividing both sides by Cos θ)

or $2\,\text{Sin}\,\theta=4(1-\text{Sin}^2\theta)-3$

or $4\,\text{Sin}^2\theta+2\,\text{Sin}\,\theta-1=0$

$\therefore$ $\text{Sin}\,\theta=\frac{-2\pm\sqrt{4+16}}{8}$

$=\frac{-2\pm2\sqrt{5}}{8}$

$=\frac{\pm\sqrt{5}-1}{4}$

We will take $\text{Sin}\,\theta=\frac{\sqrt{5}-1}{4}$ because $\theta=18°$, is an acute angle and is, therefore, *positive*.

Thus $\text{Sin}\,18°=\frac{\sqrt{5}-1}{4}$

Also, $\text{Cos}\,18°=\sqrt{1-\text{Sin}^2 18°}=\sqrt{1-\left(\frac{\sqrt{5}-1}{4}\right)^2}$

$$=\sqrt{1-\frac{5+1-2\sqrt{5}}{16}}=\sqrt{\frac{16-6+2\sqrt{5}}{16}}$$

$$=\sqrt{\frac{10+2\sqrt{5}}{16}}=\sqrt{\frac{10+2\sqrt{5}}{4}}$$

The remaining circular functions can now easily be found from these two ratios.

Again, $\text{Sin } 72^\circ = \text{Sin } (90^\circ - 18^\circ)$

$$\text{Cos } 18^\circ = \frac{\sqrt{10+2\sqrt{5}}}{4}$$

and $\text{Cos } 72^\circ = \text{Cos } (90^\circ - 18^\circ)$

$$= \text{Sin } 18^\circ = \frac{\sqrt{5}-1}{4}$$

Sin 72° and Cos 72° having thus been found we can easily find other ratios such as tan 72°, Cot 72°, etc.,

4.6.2. Trigonometric functions of 36° and 54°.

Let $\qquad \theta = 36^\circ$, So that $5\theta = 180^\circ$

or $\qquad 2\theta = 180^\circ - 3\theta$

or $\qquad \text{Sin } 2\theta = \text{Sin } (180^\circ - 3\theta) = \text{Sin } 3\theta$

or $\qquad 2 \text{ Sin } \theta \text{ Cos } \theta = 3 \text{ Sin } \theta - 4 \text{ Sin}^3 \theta$

Dividing both sides by Sin θ, we have

$$2 \text{ Cos } \theta = 3 - 4 \text{ Sin}^2 \theta$$

$$= 3 - 4(1 - \text{Cos}^2 \theta)$$

or $\qquad 4 \text{ Cos}^2 \theta - 2 \text{ Cos } \theta - 1 = 0$

$\therefore \qquad \text{Cos } \theta = \dfrac{2 \pm \sqrt{4+16}}{8}$

$$= \frac{2 \pm 2\sqrt{5}}{8} = \frac{\pm\sqrt{5}+1}{4}$$

Now $\theta = 36^\circ$, is an acute angle, therefore, we take $\text{Cos } \theta = \dfrac{\sqrt{5}+1}{4}$, the negative sign having been rejected as such.

Now $\text{Sin } 36° = \sqrt{1-\text{Cos}^2 36°} = \sqrt{1-\left(\frac{\sqrt{5}+1}{4}\right)^2}$

$$= \sqrt{1-\frac{5+1+2\sqrt{5}}{16}} = \sqrt{\frac{10-2\sqrt{5}}{16}}$$

$$= \frac{\sqrt{10-2\sqrt{5}}}{4}.$$

With the help of Sin 36° and Cos 36°, we can find the remaining trigonometric functions like tan 36°, Sec 36° *etc.*

Again angles 36° and 54° being complementary we have :

$$\text{Sin } 54° = \text{Sin } (90° - 36°) = \text{Cos } 36°$$

$$= \frac{\sqrt{5}+1}{4}$$

and $\text{Cos } 54° = \text{Cos } (90° - 36°) = \text{Sin } 36°$

$$= \frac{\sqrt{10-2\sqrt{5}}}{4}$$

EXERCISE VI

1. Find the values of Sin 18° and Cos 18°.

(*K. U. Pre. 1962*)

2. Prove that (*i*) $\text{Sin}^2\ 72° - \text{Sin}^2\ 60° = \frac{\sqrt{5}-1}{8}$.

(*ii*) $\text{Sin}^2\ 36°.\ \text{Sin}^2\ 72°.\ \text{Sin}^2\ 108°.\ \text{Sin}^2\ 144°$

$$= \frac{5}{16}.$$

3. Given $\text{Cos } 135° = -\frac{1}{\sqrt{2}}$, find Sin $67\frac{1}{2}°$ and Cos $67\frac{1}{2}°$.

4. Find the values of Sin $22\frac{1}{2}°$, Cos $22\frac{1}{2}°$, and tan $22\frac{1}{2}°$.

5. Given $\text{Sin } 60° = \frac{\sqrt{3}}{2}$, deduce the values of Sin 30° and Cos 30°.

6. If $\tan\frac{\theta}{2}=\sqrt{\frac{1-e}{1+e}}\ \tan\frac{\varphi}{2}$ show that

$$\text{Cos}\ \varphi=\frac{\text{Cos}\ \theta-e}{1-e\ \text{Cos}\theta}$$ (*K. U. Inter., 1962*)

7. If θ is an acute angle and $\text{Sin}\ \theta=\frac{2ab}{a^2+b^2}$, find $\tan\frac{\theta}{2}$.

8. Show that Cos $36°$ and Sin $18°$ are the roots of the equation $4x^2-2\sqrt{5}\ x+1=0$.

9. Find tan $\frac{A}{2}$, Sin $\frac{A}{2}$, and Cos $\frac{A}{2}$, if tan $A=\frac{21}{20}$ and $\frac{A}{2}$ lies in the first quadrant.

10. Find the value of tan $15°$, from the equation, $3\tan\theta-\tan^3\theta=1-\tan^2\theta$.

Hint :– In $\tan 3\theta=\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}$, put $\theta=15°$ and simplify the resultant equation.

CHAPTER V

The sum and Product Formulae

5·1 We already know that :—

$\text{Sin}\,(A+B) = \text{Sin}\,A\ \text{Cos}\,B + \text{Cos}A\ \text{Sin}\,B \qquad \ldots\ldots(i)$

$\text{Sin}\,(A-B) = \text{Sin}\,A\ \text{Cos}\,B - \text{Cos}\,A\ \text{sin}\,B \qquad \ldots\ldots(ii)$

Adding and subtracting these two, we get

$\text{Sin}\,(A+B) + \text{Sin}\,(A-B) = 2\ \text{Sin}\,A\ \text{Cos}\,B \qquad \ldots\ldots(iii)$

and $\text{Sin}\,(A+B) - \text{Sin}\,(A-B) = 2\ \text{Cos}\,A\ \text{Sin}\,B \qquad \ldots\ldots(iv)$

Let $A+B=P$ and $A-B=Q$

$$\therefore \quad A = \frac{P+Q}{2} \text{ and } B = \frac{P-Q}{2}$$

Hence from (iii) and (iv), we get

$$\text{Sin}\,P + \text{Sin}\,Q = 2\ \text{Sin}\frac{P+Q}{2}\,\text{Cos}\,\frac{P-Q}{2}$$

$$\text{and Sin}\,P - \text{Sin}\,Q = 2\ \text{Cos}\frac{P+Q}{2}\ \text{Sin}\ \frac{P-Q}{2}$$

5.1.2. Again, we know that :—

$\text{Cos}\,(A+B) = \text{Cos}\,A\ \text{Cos}\,B - \text{Sin}\,A\ \text{Sin}\,B \qquad \ldots\ldots(v)$

$\text{Cos}\,(A-B) = \text{Cos}\,A\ \text{Cos}\,B + \text{Sin}\,A\ \text{Sin}\,B \qquad \ldots\ldots(vi)$

Adding and subtracting these two, we get

$\text{Cos}\,(A-B) + \text{Cos}(A+B) = 2\ \text{Cos}\,A\ 2\ \text{Cos}\,B \qquad \ldots\ldots(vii)$

and $\text{Cos}\,(A+B) - \text{Cos}\,(A+B) = 2\ \text{Sin}\,A\ \text{Sin}\,B \qquad \ldots\ldots(viii)$

Let $A+B=P$ and $A-B=Q$

$$\text{So that } A = \frac{P+Q}{2} \text{ and } B = \frac{P-Q}{2}$$

Hence from (vii) and $(viii)$, we have

$$\text{Cos } P + \text{Cos } Q = 2 \text{ Cos} \frac{P+Q}{2} \text{ Cos} \frac{P-Q}{2}$$

and $$\text{Cos } Q - \text{Cos } P = 2 \text{ Sin} \frac{P+Q}{2} \text{ Sin} \frac{P-Q}{2}$$

5·2 Writing (*iii*), (*iv*) and (*vii*) and (*viii*) in the reverse order, we have

$$2 \text{ Sin } A \text{ Cos } B = \text{Sin } (A+B) + \sin (A-B)$$

$$2 \text{ Cos } A \sin B = \sin (A+B) - \text{Sin } (A-B)$$

$$2 \text{ Cos } A \text{ Cos } B = \text{Cos } (A+B) + \text{Cos } (A-B)$$

and $$2 \text{ Sin } A \text{ Sin } B = \text{Cos } (A-B) - \text{Cos } (A+B)$$

Very Important Note :—

In the above three articles, we have derived the following eight formulae. These are extremely important and the student is advised to master them as thoroughly as possible. Of these, the first four will enable the student to transform sum or difference into product, whereas the last four will enable him to transform product into sum or difference.

1. $\text{Sin } P + \text{Sin } Q = 2 \text{ Sin} \frac{P+Q}{2} \text{ Cos} \frac{P-Q}{2}$

2. $\text{Sin } P - \text{Sin} Q = 2 \text{ Cos} \frac{P+Q}{2} \text{ Sin} \frac{P-Q}{2}$

3. $\text{Cos } P + \text{Cos } Q = 2 \text{ Cos} \frac{P+Q}{2} \text{ Cos} \frac{P-Q}{2}$

4. $\text{Cos } Q - \text{Cos } P = 2 \text{Sin} \frac{P+Q}{2} \text{ Sin} \frac{P-Q}{2}$

5. $2 \text{ Sin } A \text{ Cos } B = \text{Sin } (A+B) + \text{Sin } (A-B)$
6. $2 \text{ Cos } A \text{ Sin } B = \text{Sin } (A+B) - \text{Sin } (A-B)$
7. $2 \text{ Cos } A \text{ Cos } B = \text{Cos } (A+B) + \text{Cos } (A-B)$
8. $2 \text{ Sin } A \text{ Sin } B = \text{Cos } (A-B) - \text{Cos } (A+B)$

Note :—(*i*) *In formulae 5–8, A is greater than B.*

(ii) The student is advised to commit all these formulae to memory. Of these the last four require special attention, as it been seen that students lack proper understanding of these as a result of which they commit blunders in degree classes as well.

Solved Examples

Ex. 1. Express Cos $5\theta-$Cos 7θ as a product.

Sol. Here we have to make use of formula No : (4), and in place of Q we have 5θ and in place of P we have 7Q.

$$\therefore \text{Cos } 5\theta - \text{Cos } 7\theta = 2 \text{ Sin } \frac{7\theta+5\theta}{2} \text{ Sin } \frac{7\theta-5\theta}{2}$$

$$= 2 \text{ Sin } 6\theta \text{ Sin } \theta$$

Ex. 2. Express Sin 5θ as a sum or difference.

Sol. $\text{Sin } 3\theta \text{ Sin } 5\theta = \frac{1}{2}[2 \text{ Sin } 3\theta \text{ Sin } 5\theta]$

$= \frac{1}{2}[2 \text{ Sin } 5\theta \text{ Sin } 3\theta]$

(Please note these two steps)

Here we have to make use of formula No : (8) In place of A, we have 50 and in place of B, we have 3θ.

$\therefore \frac{1}{2}[2 \text{ Sin } 5\theta \text{ Sin } 3\theta] = \frac{1}{2}[\text{Cos } (5\theta-3\theta) - \text{Cos}(5\theta+3\theta)]$

$= \frac{1}{2}[\text{Cos } 2\theta - \text{Cos } 8\theta]$

Ex. 3. Express Cos $11°+$Sin $11°$ as a product

Sol. Sin $11° =$ Sin $(90°-79°)$

$=$ Cos $79°$ $\quad [\because \text{Sin } (90°-\theta) = \text{Cos } \theta]$

(Please note this step)

$\therefore$ Cos $11°+$Sin $11°=$Cos $11°+$Cos 79

Here formula (3) is applicable.

Instead of P, we have $79°$ and instead of Q, we have 11

$$\therefore \text{Cos } 79° + \text{Cos } 11° = 2 \text{ Cos } \frac{79°+11°}{2} \text{ Cos } \frac{79°-11°}{2}$$

$$= 2 \text{ Cos } 45° \text{ Cos } 34°$$

Note :—Before putting the expression into the product form, we have to express either *Sine* into *Cosine* or *Cosine* into *Sine*.

Ex. 4. Prove that Cos 20°. Cos 30°. Cos 40°.Cos 80°

$$=\frac{\sqrt{3}}{16}$$

Sol. L. H. S.

$=\text{Cos } 20°.\ \text{Cos } 30°.\ \text{Cos } 40°.\ \text{Cos } 80°$

$=\frac{\sqrt{3}}{2}.\ \text{Cos } 20°.\ \text{Cos } 40°.\ \text{Cos } 80°$

$\left(\because \text{Cos } 30°=\frac{\sqrt{3}}{2}\right)$

$=\frac{\sqrt{3}}{4}(2\ \text{Cos } 40°\ \text{Cos } 20°)\ \text{Cos } 80°$

(*Please note this step*)

$=\frac{\sqrt{3}}{4}(\text{Cos } 60°+\text{Cos } 20°)\ \text{Cos } 80°$

[Applying formula no : 7]

$=\frac{\sqrt{3}}{4}[\frac{1}{2}\ \text{Cos } 80°+\text{Cos } 20°\ \text{Cos } 80°]$

$(\because \text{Cos } 60=\frac{1}{2})$

$=\frac{\sqrt{3}}{8}(\text{Cos } 80°+2\ \text{Cos } 80°\ \text{Cos } 20°)$

$=\frac{\sqrt{3}}{8}(\text{Cos } 80°+\text{Cos } 100°+\text{Cos } 60°)$

$=\frac{\sqrt{3}}{8}(\text{Cos } 80°+\text{Cos } 100°+\frac{1}{2})$

$=\frac{\sqrt{3}}{8}(2\ \text{Cos } 90°\ \text{Cos } 10°+\frac{1}{2})$

(Putting Cos 100°+Cos 80° into products form)

$=\frac{\sqrt{3}}{16}=$R. H. S. $(\because \text{Cos } 90°=0)$

Ex. 5. Show that $\dfrac{\text{Cos }\theta-\text{Cos }3\theta}{\text{Sin }3\theta+\text{Sin }\theta}=\tan\theta$

Sol. L. H. S.

$$=\frac{\text{Cos }\theta-\text{Cos }3\theta}{\text{Sin }3\theta+\text{Sin }\theta}=\frac{2\sin\dfrac{\theta+3\theta}{2}\text{ Sin }\dfrac{3\theta-\theta}{2}}{2\text{ Sin}\dfrac{3\theta+\theta}{2}\text{ Cos }\dfrac{3\theta-\theta}{2}}$$

[Apply formula No. 4 for the numerator and formula No. (1) for the denominator]

$$=\frac{2\text{ Sin }2\theta\text{ Sin }\theta}{2\text{ Sin }2\theta\text{ Cos }\theta}=\frac{\text{Sin }\theta}{\text{Cos }\theta}$$

$$=\tan\theta=\text{R. H. S.}$$

Ex. 6. Show that $\dfrac{\text{Sin}^2\text{ A}-\text{Sin}^2\text{ B}}{\text{Sin A Cos A}-\text{Sin B Cos B}}=\tan\text{ (A+B)}$

(K. U.)

$$\text{L. H. S.}=\frac{\text{Sin}^2\text{ A}-\text{Sin}^2\text{ B}}{\text{Sin A Cos A}-\text{Sin B Cos B}}$$

$$=\frac{2(\text{Sin}^2\text{ A}-\text{Sin}^2\text{ B})}{2\text{ Sin A Cos A}-2\text{ Sin B Cos B}}$$

(Please note this step)

$$=\frac{2\text{ Sin (A+B) Sin (A}-\text{B)}}{\text{Sin 2 A}-\text{Sin 2 B}}$$

$(\because 2\text{ Sin A Cos A}=\text{Sin 2A etc.,})$

$$=\frac{2\text{ Sin (A+B) Sin (A}-\text{B)}}{2\text{ Cos (A+B) Sin (A}-\text{B)}}=\frac{\text{Sin (A+B)}}{\text{Cos (A+B)}}$$

$$=\tan\text{ (A+B)}=\text{R. H. S.}$$

Ex. 7. Prove that

$$\frac{\text{Sin }\theta+\text{Sin }3\theta+(\text{Sin }5\theta+\text{Sin }7\theta)}{\text{Cos }\theta+\text{Cos }3\theta+\text{Cos }5\theta+\text{Cos }7\theta}=\tan 4\theta.$$

Sol. $\text{L. H. S.}=\dfrac{(\text{Sin }7\theta+\text{Sin }\theta)+(\text{Sin }5\theta+\text{Sin }3\theta)}{(\text{Cos }7\theta+\text{Cos }\theta)+(\text{Cos }5\theta+\text{Cos }3\theta)}$

$$= \frac{\left(2 \operatorname{Sin} \frac{7\theta+\theta}{2} \operatorname{Cos} \frac{7\theta-\theta}{2}\right)}{\left(2 \operatorname{Cos} \frac{7\theta+\theta}{2} \operatorname{Cos} \frac{7\theta-\theta}{2}\right)}$$

$$\frac{+\left(\left(2 \operatorname{Sin} \frac{5\theta+3\theta}{2} \operatorname{Cos} \frac{5\theta-3\theta}{2}\right)\right.}{+\left(2 \operatorname{Cos} \frac{5\theta+3\theta}{2} \operatorname{Cos} \frac{5\theta-3\theta}{2}\right)}$$

$$= \frac{2 \operatorname{Sin} 4\theta \operatorname{Cos} 3\theta + 2 \operatorname{Sin} 4\theta \operatorname{Cos}\theta}{2 \operatorname{Cos} 4\theta \operatorname{Cos} 3\theta + 2 \operatorname{Cos} 4\theta \operatorname{Cos}\theta}$$

$$= \frac{2 \operatorname{Sin} 4\theta\,(\operatorname{Cos} 3\theta + \operatorname{Cos}\theta)}{2 \operatorname{Cos} 4\theta\,(\operatorname{Cos} 3\theta + \operatorname{Cos}\theta)}$$

$$= \frac{\operatorname{Sin} 4\theta}{\operatorname{Cos} 4\theta} = \tan 4\theta = \text{R. H. S.}$$

Ex. 8. If $\operatorname{Sin}\theta = n \operatorname{Sin}(\theta+2\alpha)$

Show that $\tan(\theta+\alpha) = \frac{1+n}{1-n}\tan\alpha$

Sol. We have $\operatorname{Sin}\theta = n \operatorname{Sin}(\theta+2\alpha)$

$$\therefore \frac{\operatorname{Sin}(\theta+2\alpha)}{\operatorname{Sin}\theta} = \frac{1}{n}$$

By Componendo—Dividendo, we have

$$\frac{\operatorname{Sin}(\theta+2\alpha) + \operatorname{Sin}\theta}{\operatorname{Sin}(\theta+2\alpha) - \operatorname{Sin}\theta} = \frac{1+n}{1-n} \qquad \ldots(i)$$

Now L. H. S. $= \frac{\operatorname{Sin}(\theta+2\alpha) + \operatorname{Sin}\theta}{\operatorname{Sin}(\theta+2\alpha) - \operatorname{Sin}\theta}$

$$= \frac{2 \operatorname{Sin} \frac{\theta+2\alpha+\theta}{2} \operatorname{Cos} \frac{\theta+2\alpha-\theta}{2}}{2 \operatorname{Cos} \frac{\theta+2\alpha+\theta}{2} \operatorname{Sin} \frac{\theta+2\alpha-\theta}{2}}$$

$$= \frac{2 \operatorname{Sin}(\theta+\alpha) \operatorname{Cos}\alpha}{2 \operatorname{Cos}(\theta+\alpha) \operatorname{Sin}\alpha} = \tan(\theta+\alpha) \operatorname{Cot}\alpha$$

$\therefore$ From (i) we get

$$\tan(\theta+\alpha)\text{ Cot }\alpha=\frac{1+n}{1-n}$$

$$\therefore \tan(\theta+\alpha)=\frac{1+n}{1-n}\tan\alpha.$$

Ex. 9. Show that :—

$$\text{Cos }(36°-A)\text{ Cos }(36°+A)+\text{Cos }(54°+A)(\text{Cos }54°-A)=\text{Cos }2A$$

Sol. L. H. S.

$$\tfrac{1}{2}[2\text{ Cos }(36°+A)\text{ Cos }(36°-A)+2\text{ Cos }(54°+A)\times(\text{Cos }54°-A)]$$

Applying formula no. (7), we get :—

$$=\tfrac{1}{2}[(\text{Cos }72°+\text{Cos }2A)+(\text{Cos }108°+\text{Cos }2A)]$$

$$=\tfrac{1}{2}[(\text{Cos }72°+\text{Cos }2A)+\{\text{Cos }(180°-72°)+\text{Cos }2A\}]$$

(Please note this step)

$$=\tfrac{1}{2}[(\text{Cos }72°+\text{Cos }2A)+(-\text{Cos }72°+\text{Cos }2A)]$$

$(\because \text{Cos }(\pi-\theta)=-\text{Cos }\theta)$

$$=\tfrac{1}{2}\{2\text{ Cos }2A\}=\text{Cos }2A=\text{R. H. S.}$$

EXERCISE VII

(1) Express the following in the product form :—

(*i*) $\text{Sin }3\theta+\text{Sin }\theta$ (*ii*) $\text{Sin }6\theta-\text{Sin }4\theta$

(*iii*) $\text{Cos }2\theta+\text{Cos }8\theta$ (*iv*) $\text{Cos }\theta-\text{Cos }5\theta$

(*v*) $\text{Cos }3A-\text{Cos }7A$.

(2) Express the following to the sum form :—

(*i*) $\text{Cos }20°.\text{ Cos }40°$ (*ii*) $\text{Sin }11A.\text{ Sin }A$

(*iii*) $\text{Cos }7A.\text{ Sin }3A$ (*iv*) $2\text{ Sin }7A.\text{ Sin }3A$

(*v*) $\text{Sin }8A.\text{ Sin }4A$ (*vi*) $\text{Sin }7A.\text{ Cos }3A$

(3) Prove that :—

(*i*) $\text{Sin }51°+\text{Cos }81°=\text{Cos }21°$

(*ii*) $\text{Sin }47°+\text{Cos }77°=\text{Cos }17°$

(*iii*) $\text{Cos }17°-\text{Cos }77°=\text{Sin }47°$

Prove the following :—

(4) $\dfrac{\text{Sin } \alpha+\text{Sin } \beta}{\text{Cos } \alpha+\text{Cos } \beta}=\tan\dfrac{\alpha+\beta}{2}$

(5) $\dfrac{\text{Sin } \alpha-\text{Sin } \beta}{\text{Cos } \alpha+\text{Cos } \beta}=\tan\dfrac{\alpha-\beta}{2}$

(6) $\dfrac{\text{Sin } \beta+\text{Sin } \alpha}{\text{Cos } \beta-\text{Cos } \alpha}=\text{Cot } \dfrac{\alpha-\beta}{2}$

(7) $\dfrac{\text{Sin } \alpha-\text{Sin } \beta}{\text{Cos } \beta-\text{Cos } \alpha}=\text{Cot } \dfrac{\alpha+\beta}{2}$

(8) $\dfrac{\text{Sin } \alpha-\text{Sin } \beta}{\text{Sin } \alpha+\text{Sin } \beta}=\dfrac{\tan \dfrac{\alpha-\beta}{2}}{\tan \dfrac{\alpha+\beta}{2}}$

(9) $\dfrac{\text{Cos } \beta-\text{Cos } \alpha}{\text{Cos } \beta+\text{Cos } \alpha}=\tan\dfrac{\alpha+\beta}{2}\tan\dfrac{\alpha-\beta}{2}$

(10) $\dfrac{\text{Cos } 9^\circ+\text{Sin } 9^\circ}{\text{Cos } 9^\circ-\text{Sin } 9^\circ}=\tan 54^\circ$

(11) Show that $\dfrac{\text{Sin}^2 A-\text{Sin}^2 B}{\text{Sin } A \text{ Cos } A-\text{Sin } B \text{ Cos } B}=\tan (A+B)$

(K.U. Inter. 1960)

Prove that :—

(12) $\dfrac{\text{Sin } 7A+\text{Sin } 3A}{\text{Cos } 7A+\text{Cos } 3A}=\tan 5A$

(13) $\dfrac{\text{Cos } A-\text{Cos } 3A}{\text{Sin } 3A-\text{Sin } A}=\tan 2A$

(14) $\dfrac{\text{Cos } 7A-\text{Cos } 9A}{\text{Sin } 9A-\text{Sin } 7A}=\tan 8A$

(15) $\dfrac{\text{Sin } A+2 \text{ Sin } 3A+\text{Sin } 5A}{\text{Sin } 3A+2 \text{ Sin } 5A+\text{Sin } 7A}=\dfrac{\text{Sin } 3A}{\text{Sin } 5A}$

(16) $\dfrac{\text{Sin } A+\text{Sin } (A+B)+\text{Sin } (A+2B)}{\text{Cos } A+\text{Cos } (A+B)+\text{Cos } (A+2B)}=\tan (A+B)$

(17) $\dfrac{\text{Sin }\theta+\text{Sin }2\,\theta+\text{Sin }4\,\theta+\text{Sin }5\,\theta}{\text{Cos }\theta+\text{Cos }2\,\theta+\text{Cos }4\,\theta+\text{Cos }5\,\theta}=\tan 3\,\theta$

(18) $\dfrac{\text{Sin }4\,\theta+2\text{ Sin }3\,\theta+\text{Sin }2\,\theta}{\text{Cos }2\,\theta-\text{Cos }4\,\theta}=\text{Cot }\dfrac{\theta}{2}$

(19) $\dfrac{\text{Sin }3\,\theta-\text{Sin }\theta}{\text{Cos }3\,\theta+2\text{ Cos }2\,\theta+\text{Cos }\theta}=\tan\dfrac{\theta}{2}$

Prove that :—

(20) $\text{Cos }(A+B)+\text{Sin }(A-B)=2\text{ Sin }(45^\circ+A)\times\text{Cos }(45^\circ+B)$

(**Hint** :—Put the R. H. S. into *sum* form)

(21) $\text{Cos }(36^\circ-A)\text{ Cos }(36^\circ+A)+\text{Cos }(54^\circ+A)\text{ Cos }(54^\circ-A)=\text{Cos }2A$

(22) $\text{Cos A Cos B}=\text{Cos}^2\dfrac{A-B}{2}-\text{Sin}^2\dfrac{A+B}{2}$

(23) $\dfrac{\text{Cos 2A. Cos 3A}-\text{Cos 2A. Cos 7A}+\text{Cos A Cos 10A}}{\text{Sin 4A Sin 3A}-\text{Sin 2A Sin 5A}+\text{Sin 4A Sin 7A}}=\text{Cot 6A Cot 5A}$

(24) $\dfrac{\text{Sin A}+\text{Sin 3A}}{\text{Cos A}+\text{Cos 3A}}+\dfrac{\text{Sin 2A}+\text{Sin 4A}}{\text{Cos 2A}+\text{Cos 4A}}=\dfrac{\text{Sin 5A}}{\text{Cos 2A Cos 3A}}$

(25) If $A+B+C+D=180^\circ$, prove that

$\text{Cos 2A}-\text{Cos 2B}+\text{Cos 2C}-\text{Cos 2D}$
$=4\text{ Sin }(A+B)\text{ Sin }(B+C)\text{ Cos }(C+A)$

(26) prove that :—

(i) $\text{Cos }20^\circ.\ \text{Cos }40^\circ.\ \text{Cos }80^\circ=\frac{1}{8}$

(ii) $\text{Sin }20^\circ.\ \text{Sin }40^\circ.\ \text{Sin }80^\circ.\ \text{Sin }90^\circ=\dfrac{\sqrt{3}}{8}$ (*P.U. 1947*)

(iii) $\text{Cos }20^\circ.\ \text{Cos }40^\circ.\ \text{Cos }60^\circ.\ \text{Cos }80^\circ=\frac{1}{16}$ (*P.U. 1948*)

(iv) $\text{Sin }20^\circ.\ \text{Sin }40^\circ.\ \text{Sin }60^\circ.\ \text{Sin }80^\circ=\frac{3}{16}$

(v) $\text{Cos }20^\circ.\ \text{Cos }30^\circ.\ \text{Cos }40^\circ.\ 80^\circ=\dfrac{\sqrt{3}}{16}$

(*P. U. 1951*)

(vi) $\text{Sin }\dfrac{\pi}{5}\text{ Sin }\dfrac{2\pi}{5}.\ \text{Sin }\dfrac{3\pi}{5}.\ \text{Sin }\dfrac{4\pi}{5}\ \ \dfrac{5}{16}$

(vii) $\text{Cos } 36°. \text{Cos } 72°. \text{Cos } 108°. \text{ Cos } 144° = \frac{1}{16}$

(27) Prove that :—

(i) $(\text{Cos } \alpha + \text{Cos } \beta)^2 + (\text{Sin } \alpha - \text{Sin } \beta)^2 = 4 \text{ Cos}^2 \frac{\alpha+\beta}{2}$

(ii) $(\text{Cos } \alpha - \text{Cos } \beta)^2 + (\text{Sin } \alpha - \text{Sin } \beta)^2 = 4 \text{ Sin}^2 \frac{\alpha-\beta}{2}$

(28) If $\text{Cos } (A-B) = 3 \text{ Cos } (A+B)$, prove that $\text{Cot } A. \text{Cot } B = 2$

(29) Prove that :—

$\text{Cos } \alpha + \text{Cos } \beta + \text{Cos } \gamma + \text{Cos } (\alpha+\beta+\gamma)$

$= 4 \text{ Cos } \frac{\alpha+\beta}{2} \text{Cos } \frac{\beta+\gamma}{2} \text{Cos } \frac{\gamma+\alpha}{2}$

(30) If $\frac{\text{Cos } (\alpha+\beta)}{\text{Cos } (\alpha-\beta)} = \frac{\text{Sin } (\gamma-\delta)}{\text{Sin } (\gamma+\delta)}$

Show that $\tan \alpha. \tan \beta. \tan \gamma = \tan\delta$

CHAPTER VI

Trigonometrical Identities and Eliminations.

6.1. Identities. If A, B, C denote the angles of a triangle ABC, then $A+B+C=180^\circ$.

A number of identical relations hold between the trigonometrical ratios of the angles. The following examples will illustrate the methods employed in to proving these identities.

6.2. *Identities holding between Sines of three angles.*

Ex. 1. If $A+B+C=180^\circ$, prove that $\text{Sin } 2A+\text{Sin } 2B+\text{Sin } 2C=4\text{ Sin A Sin B Sin C}$ *(K.U. 1955)*

Sol. L.H.S. $=\text{Sin } 2A+\text{Sin } 2B+\text{Sin } 2C$

$$=2\text{ Sin }\frac{2A+2B}{2}\text{ Cos }\frac{2A-2B}{2}+2\text{ Sin C Cos C}$$

$$=2\text{ Sin }(A+B)\text{ Cos }(A-B)+2\text{ Sin C Cos C}$$

$$=2\text{ Sin C Cos }(A-B)+2\text{ Sin C Cos C}$$

$[\because A+B=180^\circ-C \quad \therefore \text{Sin } (A+B)=\text{Sin } (180^\circ-C)=\text{Sin C}]$

$$=2\text{ Sin C}[\text{Cos}(A-B)+\text{Cos C}]$$

$$=2\text{ Sin C}[\text{Cos}(A-B)+\text{Cos}(180^\circ-\overline{A+B})]$$

$(\because C=180^\circ-\overline{A+B})$

$$=2\text{ Sin C}[\text{Cos }(A-B)-\text{Cos}(A+B)]$$

$[\because \text{Cos }(180^\circ-\overline{A+B})=-\text{Cos }(A+B)]$

$$=2\text{ Sin C}\left[2\text{ Sin}\frac{A-B+A+B}{2}\text{ Sin }\frac{A+B-A+B}{2}\right]$$

$$=2\text{ Sin C}.\ 2\text{ Sin A}.\ \text{Sin B}=4\text{ Sin A Sin B Sin C}$$

$=$ R.H.S.

Note :—The student is advised to commit the above result to memory, as many interesting results can be dervied from it. Some of them are explained below: —

We have seen that

$$\text{Sin } 2A+\text{Sin}2\ B+\text{Sin } 2C=4 \text{ Sin } A \text{ Sin } B \text{ Sin } C$$

(*i*) Replacing A, B, C by $\frac{\pi}{2}-\frac{A}{2}$, $\frac{\pi}{2}-\frac{B}{2}$, $\frac{\pi}{2}-\frac{C}{2}$ respectively, we get

$$\text{Sin } 2\left(\frac{\pi}{2}-\frac{A}{2}\right)+\text{Sin } 2\left(\frac{\pi}{2}-\frac{B}{2}\right)+\text{Sin } 2\left(\frac{\pi}{2}-\frac{C}{2}\right)$$

$$=4 \text{ Sin}\left(\frac{\pi}{2}-\frac{A}{2}\right) \text{Sin}\left(\frac{\pi}{2}-\frac{B}{2}\right) \text{Sin}\left(\frac{\pi}{2}-\frac{C}{2}\right)$$

$$\text{or Sin}(\pi-A)+\text{Sin}(\pi-B)+\text{Sin}(\pi-C)=4 \text{ Sin}\left(\frac{\pi}{2}-\frac{A}{2}\right)$$

$$\times\text{Sin}\left(\frac{\pi}{2}-\frac{B}{2}\right)\text{Sin}\left(\frac{\pi}{2}-\frac{C}{2}\right)$$

$$\text{or Sin } A+\text{Sin } B+\text{Sin } C=4 \text{ Cos}\frac{A}{2} \text{ Cos}\frac{B}{2} \text{ Cos}\frac{C}{2}$$

$$\left[\because \text{Sin } (\pi-\theta)=\text{Sin } \theta \text{ and Sin}\left(\frac{\pi}{2}-\theta\right)=\text{Cos}\theta\right]$$

(*ii*) Similarly, changing A,B,C, into $\pi-2A, \pi-2B, \pi-2C$ respectively, we have

$$\text{Sin } 2(\pi-2A)+\text{Sin } 2(\pi-2B)+\text{Sin } 2(\pi-2C)$$

$$=4 \text{ Sin } (\pi-2A) \text{ Sin } (\pi-2B)\text{Sin } (\pi-2C)$$

$$\text{or Sin } (2\pi-4A)+\text{Sin}(2\pi-4B)+\text{Sin } (2\pi-4C)$$

$$=4 \text{ Sin } (\pi-2A) \text{ Sin } (\pi-2B) \text{ Sin } (\pi-2C)$$

$$\therefore \text{Sin } 4A+\text{Sin } 4B+\text{Sin } 4C=4 \text{ Sin } 2A \text{ Sin } 2B \text{ Sin } 2C$$

Ex. 2. If $A+B+C=180°$, prove that

$$\operatorname{Sin} A+\operatorname{Sin} B+\operatorname{Sin} C=4 \operatorname{Cos}\frac{A}{2}\operatorname{Cos}\frac{B}{2}\operatorname{Cos}\frac{C}{2}$$

(P.U. 1948)

Sol. L. H. S.

$$=(\operatorname{Sin} A+\operatorname{Sin} B)+\operatorname{Sin} C$$

$$=2\operatorname{Sin}\frac{A+B}{2}\operatorname{Cos}\frac{A-B}{2}+2\operatorname{Sin}\frac{C}{2}\operatorname{Cos}\frac{C}{2}$$

$$=2\operatorname{Sin}\left(90°-\frac{C}{2}\right)\operatorname{Cos}\frac{A-B}{2}+2\operatorname{Sin}\frac{C}{2}\operatorname{Cos}\frac{C}{2}$$

$$\left(\because \frac{A+B}{2}=90°-\frac{C}{2}\right)$$

$$=2\operatorname{Cos}\frac{C}{2}\operatorname{Cos}\frac{A-B}{2}+2\operatorname{sin}\frac{C}{2}\operatorname{Cos}\frac{C}{2}$$

$$=2\operatorname{Cos}\frac{C}{2}\left(\operatorname{Cos}\frac{A-B}{2}+\operatorname{Sin}\frac{C}{2}\right)$$

$$=2\operatorname{Cos}\frac{C}{2}\left[\operatorname{Cos}\frac{A-B}{2}+\operatorname{Sin}\left(90-\frac{A+B}{2}\right)\right]$$

$$=2\operatorname{Cos}\frac{C}{2}\left[\operatorname{Cos}\frac{A-B}{2}+\operatorname{Cos}\frac{A+B}{2}\right]$$

$$=2\operatorname{Cos}\frac{C}{2}\left[2\operatorname{Cos}\frac{A}{2}\operatorname{Cos}\frac{B}{2}\right]$$

$$=4\operatorname{Cos}\frac{A}{2}\operatorname{Cos}\frac{B}{2}\operatorname{Cos}\frac{C}{2}=\text{R. H. S.}$$

6.3. *Identities holding between Cosines of three angles.*

Ex. 3. If $A+B+C=180°$, show that

$$\operatorname{Cos} 2A+\operatorname{Cos} 2B+\operatorname{Cos} 2C=-1-4\operatorname{Cos} A\operatorname{Cos} B\operatorname{Cos} C$$

(D.U. 1951)

Sol. L. H. S. $=(\operatorname{Cos} 2A+\operatorname{Cos} 2B)+\operatorname{Cos} 2C$

$$=2\operatorname{Cos}(A+B)\operatorname{Cos}(A-B)+(2\operatorname{Cos}^2 C-1)$$

$$=-1+2[\operatorname{Cos}(180°-C)\operatorname{Cos}(A-B)+\operatorname{Cos}^2 C]$$

$$=-1+2[-\operatorname{Cos} C\operatorname{Cos}(A-B)+\operatorname{Cos}^2 C]$$

$$[\because \operatorname{Cos}(\pi-\theta)=-\operatorname{Cos}\theta]$$

$$= -1-2\ \text{Cos}\ C[\text{Cos}\ (A-B)-\text{Cos}\ C]$$
$$= -1-2\ \text{Cos}\ C\ [\text{Cos}\ (A-B)-\text{Cos}\ (180^\circ-\overline{A+B})]$$
$$= -1-2\ \text{Cos}\ C[\text{Cos}\ (A-B)+\text{Cos}(A+B)]$$
$$= -1-2\ \text{Cos}\ C[2\ \text{Cos}\ A\ \text{Cos}\ B]$$
$$= -1-4\ \text{Cos}\ A\ \text{Cos}\ B\ \text{Cos}\ C = \text{R. H. S.}$$

Ex. 4. If $A+B+C=180^\circ$, show that

$$\text{Cos}\ A+\text{Cos}\ B+\text{Cos}\ C=1+4\ \text{Sin}\ \frac{A}{2}\ \text{Sin}\ \frac{B}{2}\ \text{Sin}\ \frac{C}{2}$$

(K. *U. 1952*)

Sol. L. H. S. $=(\text{Cos}\ A+\text{Cos}\ B)+\text{Cos}\ C$

$$=2\ \text{Cos}\ \frac{A+B}{2}\ \text{Cos}\ \frac{A-B}{2}+\left(1-2\ \text{Sin}^2\ \frac{C}{2}\right)$$

$$(\because\ \text{Cos}\ 2\theta=1-2\ \text{Sin}^2\theta)$$

$$=1+2\ \text{Cos}\left(90^\circ-\frac{C}{2}\right)\text{Cos}\ \frac{A-B}{2}-2\ \text{Sin}^2\frac{C}{2}$$

$$=1+2\ \text{Sin}\frac{C}{2}\ \text{Cos}\ \frac{A-B}{2}-2\ \text{Sin}^2\frac{C}{2}$$

$$=1+2\ \text{Sin}\frac{C}{2}\left[\text{Cos}\ \frac{A-B}{2}-\text{Sin}\frac{C}{2}\right]$$

$$=1+2\ \sin\frac{C}{2}\left[\text{Cos}\ \frac{A-B}{2}-\text{Sin}\left(90^\circ-\frac{A+B}{2}\right)\right]$$

$$=1+2\ \text{Sin}\frac{C}{2}\left(\text{Cos}\ \frac{A-B}{2}-\text{Cos}\ \frac{A+B}{2}\right)$$

$$=1+2\ \text{Sin}\ \frac{C}{2}\left(2\ \text{Sin}\ \frac{A}{2}\ \text{Sin}\ \frac{B}{2}\right)$$

$$=1+4\ \text{Sin}\frac{A}{2}\ \text{Sin}\ \frac{B}{2}\ \text{Sin}\ \frac{C}{2}=\text{R. H. S.}$$

Note :—*The student is advised to memorize the above identity.*

6.4. *Identities holding between the squares of Sines and Cosines of three angles.*

Ex. 5. If $A+B+C=180^\circ$, show that

$$\cos^2 A+\cos^2 B-\cos^2 C=1-2 \sin A \sin B \cos C$$

Sol. L. H. S$=\cos^2 A+\cos^2 B-\cos^2 C$

$=1-\sin^2 A+\cos^2 B-\cos^2 C$

$=1+(\cos^2 B-\sin^2 A)-\cos^2 C$

$=1+\cos (A+B) \cos (A-B)-\cos^2 C$

$[\because \cos (A+B) \cos (A-B)=\cos^2 B-\sin^2 A]$

$=1+\cos (180^\circ-C) \cos (A-B)-\cos^2 C$

$=1-\cos C \cos (A-B)-\cos^2 C$

$=1-\cos C [\cos (A-B)+\cos C]$

$=1-\cos C [\cos (A-B)+\cos (180^\circ-A+B)]$

$=1-\cos C [\cos (A-B)-\cos (A+B)]$

$=1-\cos C (2 \sin A. \sin B)$

$=1-2 \sin A. \sin B. \cos C$

Ex. 6. If $A+B+C=180^\circ$, Show that

$$\cos^2 \frac{A}{2}+\cos^2 \frac{B}{2}+\cos^2 \frac{C}{2}=$$

$$2\left(1+\sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}\right)$$

Sol. L. H. S.$=\cos^2 \frac{A}{2}+\cos^2 \frac{B}{2}+\cos^2 \frac{C}{2}$

$=\frac{1+\cos A}{2}+\frac{1+\cos B}{2}+\left(1-\sin^2 \frac{C}{2}\right)$

$\left(\because \cos A=2 \cos^2 \frac{A}{2}-1 \text{ etc.}\right)$

$=\left(\frac{1}{2}+\frac{1}{2}+1\right)+\frac{1}{2}\left(\cos A+\cos B\right)-\sin^2 \frac{C}{2}$

(Please note this step)

$=2+\frac{1}{2}\left(2 \cos\frac{A+B}{2} \cos \frac{A-B}{2}\right)-\sin^2 \frac{C}{2}$

$$=2+\text{Cos}\ \frac{A+B}{2}\ \text{Cos}\ \frac{A-B}{2}-\text{Sin}^2\ \frac{C}{2}$$

$$=2+\text{Cos}\left(90^\circ-\frac{C}{2}\right)\text{Cos}\ \frac{A-B}{2}-\text{Sin}^2\ \frac{C}{2}$$

$$=2+\text{Sin}\ \frac{C}{2}\cdot\text{Cos}\ \frac{A-B}{2}-\text{Sin}^2\ \frac{C}{2}$$

$$=2+\text{Sin}\ \frac{C}{2}\left[\text{Cos}\ \frac{A-B}{2}-\text{Sin}\ \frac{C}{2}\right]$$

$$=2+\text{Sin}\ \frac{C}{2}\left[\text{Cos}\ \frac{A-B}{2}\ \text{Sin}\ \left(90^\circ-\frac{A+B}{2}\right)\right]$$

$$=2+\text{Sin}\ \frac{C}{2}\left[\text{Cos}\ \frac{A-B}{2}-\text{Cos}\ \frac{A+B}{2}\right]$$

$$=2+\text{Sin}\ \frac{C}{2}\left[2\ \text{Sin}\ \frac{A}{2}\ \text{Sin}\frac{B}{2}\right]$$

$$=2\left(1+\text{Sin}\frac{A}{2}\ \text{Sin}\ \frac{B}{2}\ \text{Sin}\ \frac{C}{2}\right)=\text{R. H. S.}$$

Ex. 7. If $A+B+C=180^\circ$, prove that :—

$$\text{Sin}^2\ \frac{A}{2}+\text{Sin}^2\ \frac{B}{2}+\text{Sin}^2\ \frac{C}{2}=1-2\ \text{Sin}\ \frac{A}{2}$$

$$\text{Sin}\ \frac{B}{2}\ \text{Sin}\ \frac{C}{2}$$ (*P. U. 1949*)

Sol. $\text{L. H. S.}=\text{Sin}^2\ \frac{A}{2}+\text{Sin}^2\ \frac{B}{2}+\text{Sin}^2\ \frac{C}{2}$

$$=\frac{1-\text{Cos A}}{2}+\frac{1-\text{Cos B}}{2}+\text{Sin}^2\ \frac{C}{2}$$

($\because$ $1-\text{Cos A}=2\ \text{Sin}^2\frac{A}{2}$ *etc.*,)

$$=\left(\frac{1}{2}+\frac{1}{2}\right)-\frac{1}{2}\ (\text{Cos A}+\text{Cos B})+\text{Sin}^2\ \frac{C}{2}$$

(Please note this step)

$$=1-\tfrac{1}{2}\left(2\,\text{Cos}\,\frac{A+B}{2}\,\text{Cos}\,\frac{A-B}{2}\right)+\text{Sin}^2\,\frac{C}{2}$$

$$=1-\text{Cos}\,\frac{A+B}{2}\,\text{Cos}\,\frac{A-B}{2}+\text{Sin}^2\,\frac{C}{2}$$

$$=1-\text{Cos}\left(90^\circ-\frac{C}{2}\right)\text{Cos}\,\frac{A-B}{2}+\text{Sin}^2\,\frac{C}{2}$$

$$=1-\text{Sin}\,\frac{C}{2}\,\text{Cos}\,\frac{A-B}{2}+\text{Sin}^2\,\frac{C}{2}$$

$$=1-\text{Sin}\,\frac{C}{2}\left(\text{Cos}\,\frac{A-B}{2}-\text{Sin}\,\frac{C}{2}\right)$$

$$=1-\text{Sin}\,\frac{C}{2}\left[\text{Cos}\,\frac{A-B}{2}-\text{Sin}\left(90^\circ-\frac{A+B}{2}\right)\right]$$

$$=1-\text{Sin}\,\frac{C}{2}\left(\text{Cos}\,\frac{A-B}{2}-\text{Cos}\,\frac{A+B}{2}\right)$$

$$=1-\text{Sin}\,\frac{C}{2}\left(2\,\text{Sin}\,\frac{A}{2}\,\text{Sin}\,\frac{B}{2}\right)$$

$$=1-2\,\text{Sin}\,\frac{A}{2}\,\text{Sin}\,\frac{B}{2}\,\text{Sin}\,\frac{C}{2}=\text{R.H.S.}$$

6·5. *Identities holding between tangents or cotangents of three angles.*

Ex. 8. If $A+B+C=180^\circ$, prove that

$$\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}=1$$

(K. U. 1949)

Sol. $A+B+C=180^\circ$

$$\therefore\quad \frac{A}{2}+\frac{B}{2}=90^\circ-\frac{C}{2}$$

$$\text{or}\ \tan\left(\frac{A}{2}+\frac{B}{2}\right)=\tan\left(90^\circ-\frac{C}{2}\right)$$

$$\text{or } \frac{\tan\frac{A}{2}+\tan\frac{B}{2}}{1-\tan\frac{A}{2}\tan\frac{B}{2}}=\text{Cot}\frac{C}{2}$$

$$=\frac{1}{\tan\frac{C}{2}}$$

By Cross-Multiplication, we get

$$\tan\frac{A}{2}\tan\frac{C}{2}+\tan\frac{B}{2}\tan\frac{C}{2}=1-\tan\frac{A}{2}\tan\frac{B}{2}$$

$$\text{or } \tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}=1$$

Ex. 9. If $A+B+C=180°$, prove that

$$\text{Cot}\frac{A}{2}+\text{Cot}\frac{B}{2}+\text{Cot}\frac{C}{2}=\text{Cot}\frac{A}{2}\text{Cot}\frac{B}{2}\times\text{Cot}\frac{C}{2}$$

Sol. $\frac{A}{2}+\frac{B}{2}=90°-\frac{C}{2}$

$$\therefore\quad \text{Cot}\left(\frac{A}{2}+\frac{B}{2}\right)=\text{Cot}\left(90°-\frac{C}{2}\right)$$

$$\text{or } \frac{\text{Cot}\frac{A}{2}.\text{Cot}\frac{B}{2}-1}{\text{Cot}\frac{A}{2}+\text{Cot}\frac{B}{2}}=\tan\frac{C}{2}$$

$$\left[\because \text{Cot}(A+B)=\frac{\text{Cot A Cot B}-1}{\text{Cot A}+\text{Cot B}}\right]$$

$$=\frac{1}{\text{Cot}\frac{C}{2}}$$

By Cross-Multiplication, we get

$$\text{Cot}\frac{A}{2}\,.\,\text{Cot}\frac{B}{2}\,\text{Cot}\frac{C}{2}-\text{Cot}\frac{C}{2}=\text{Cot}\frac{A}{2}+\text{Cot}\frac{B}{2}$$

$$\therefore\ \text{Cot}\frac{A}{2}+\text{Cot}\frac{B}{2}+\text{Cot}\frac{C}{2}=\text{Cot}\frac{A}{2}\,.\,\text{Cot}\frac{B}{2}\,.\,\text{Cot}\frac{C}{2}$$

Ex. 10. Prove that

$$\frac{\text{Cot A}+\text{Cot B}}{\tan A+\tan B}+\frac{\text{Cot B}+\text{Cot C}}{\tan B+\tan C}+\frac{\text{Cot C}+\text{Cot A}}{\tan C+\tan A}=1,$$

if $A+B+C=180°$

Sol. L.H.S.

$$=\frac{\frac{1}{\tan A}+\frac{1}{\tan B}}{\tan A+\tan B}+\frac{\frac{1}{\tan B}+\frac{1}{\tan C}}{\tan B+\tan C}+\frac{\frac{1}{\tan C}+\frac{1}{\tan A}}{\tan C+\tan A}$$

$$=\frac{\frac{\tan A+\tan B}{\tan A.\ \tan B}}{\tan A+\tan B}+\frac{\frac{\tan B+\tan C}{\tan B.\ \tan C}}{\tan B+\tan C}+\frac{\frac{\tan C+\tan A}{\tan C.\ \tan A}}{\tan C+\tan A}$$

$$=\frac{1}{\tan A+\tan B}+\frac{1}{\tan B+\tan C}+\frac{1}{\tan C+\tan A}\quad\ldots\ldots(i)$$

$$=\frac{\tan C+\tan A+\tan B}{\tan A.\ \tan B.\ \tan C}$$

Now $\quad A+B=180°-C$

$\therefore\ \tan(A+B)=\tan(180°-C)$

or $\dfrac{\tan A+\tan B}{1-\tan A.\ \tan B}=-\tan C \qquad \because\ \tan(\pi-\theta)=-\tan\theta$

By cross—multiplication, we get

$\tan A+\tan B=-\tan C+\tan A\tan B\tan C$

$\therefore\ \tan A+\tan B+\tan C=\tan A.\ \tan B.\ \tan C$

Substituting this value of $\tan A+\tan B+\tan C$ in (i) we have

$$\text{L. H. S.}=\frac{\tan A.\ \tan B.\ \tan C}{\tan A.\ \tan B.\ \tan C}=1=\text{R. H. S.}$$

Ex. 11. If $x+y+z=xyz$, prove that

$$\frac{3x-x^3}{1-3x^2}+\frac{3y-y^3}{1-3y^2}+\frac{3z-z^3}{1-3z^2}=\frac{3x-x^3}{1-3x^2}\cdot\frac{3y-y^3}{1-3y^2}\cdot\frac{3z-z^3}{1-3z^2}$$

Sol. Put $x=\tan A$, $y=\tan B$, and $z=\tan C$

Now we are given that

$$x+y+z=xyz$$

or $\tan A+\tan B+\tan C-3\tan A.\tan B.\tan C=0$ (i)

Now $\tan(A+B+C)$

$$=\frac{\tan A+\tan B+\tan C-\tan A.\tan B.\tan C}{1-\tan A\tan B-\tan B\tan C-\tan C\tan A}$$

[Article 3.2 (iii)]

$=0=\tan n\pi$ ......from (i)

or $A+B+C=n\pi$

$\therefore\ 3A+3B+3C=3n\pi$

or $\tan(3A+3B+3C)=\tan 3n\pi=0$

or $\dfrac{\tan 3A+\tan 3B+\tan 3C-\tan 3A.\tan 3B.\tan 3C}{1-\tan 3A.\tan 3B-\tan 3B.\tan 3C-\tan 3C.\tan 3A}=0$

or $\tan 3A+\tan 3B+\tan 3C=\tan 3A.\tan 3B.\tan 3C$

or $\dfrac{3\tan A-\tan^3 A}{1-3\tan^2 A}+\dfrac{3\tan B-\tan^3 B}{1-3\tan^2 B}+\dfrac{3\tan C-\tan^3 C}{1-3\tan^2 C}$

$$=\frac{3\tan A-\tan^3 A}{1-3\tan^2 A}\cdot\frac{3\tan B-\tan^3 B}{1-3\tan^2 B}\cdot\frac{3\tan C-\tan^3 C}{1-3\tan^2 C}$$

Hence $\dfrac{3x-x^3}{1-3x^2}+\dfrac{3y-y^3}{1-3y^2}+\dfrac{3z-z^3}{1-3z^2}$

$$=\frac{3x-x^3}{1-3x^2}\cdot\frac{3y-y^3}{1-3y^2}\cdot\frac{3z-z^3}{1-3z^2}$$

EXERCISE VIII

(*A*) Identities holding between *Sines* of *three* angles.

If A+B+C=180°, show that :

1. $\operatorname{Sin} A+\operatorname{Sin} B+\operatorname{Sin} C=4 \operatorname{Cos}\frac{A}{2} \operatorname{Cos}\frac{B}{2} \operatorname{Cos}\frac{C}{2}$

2. $\operatorname{Sin} A+\operatorname{Sin} B-\operatorname{Sin} C=4 \operatorname{Sin}\frac{A}{2} \operatorname{Sin}\frac{B}{2} \operatorname{Cos}\frac{C}{2}$

3. $\operatorname{Sin} 2A+\operatorname{Sin} 2B+\operatorname{Sin} 2C=4 \operatorname{Sin} A \operatorname{Sin} B \operatorname{Sin} C$ (*K. U. 1955*)

4. $\operatorname{Sin} 2A-\operatorname{Sin} 2B+\operatorname{Sin} 2C=4 \operatorname{Cos} A \operatorname{Sin} B \operatorname{Cos} C$ (*K. U. 1953*)

(*B*) Identities holding between *Cosines* of *three* angles.

If A+B+C=180°, prove that :

5. $\operatorname{Cos} A+\operatorname{Cos} B+\operatorname{Cos} C=1+4 \operatorname{Sin}\frac{A}{2} \operatorname{Sin}\frac{B}{2} \operatorname{Sin}\frac{C}{2}$

6. $\operatorname{Cos} 2A+\operatorname{Cos} 2B+\operatorname{Cos} 2C$
$=-1-4 \operatorname{Cos} A \operatorname{Cos} B \operatorname{Cos} C$

7. $\operatorname{Cos} 2A+\operatorname{Cos} 2B-\operatorname{Cos} 2C$
$=1-4 \operatorname{Sin} A \operatorname{Sin} B \operatorname{Cos} C$ (*K. U. 1949*)

8. $\operatorname{Cos}\frac{A}{2}.\operatorname{Cos}\frac{B-C}{3}+\operatorname{Cos}\frac{B}{2}\operatorname{Cos}\frac{C-A}{2}+\operatorname{Cos}\frac{C}{2}.$
$\operatorname{Cos}\frac{A-B}{2}=\operatorname{Sin} A+\operatorname{Sin} B+\operatorname{Sin} C$
(*P. U. 1943*)

(*C*) Identities holding between the Squares of *Sines* and *Cosines* of three angles.

If A+B+C=180°, prove that

9. $\operatorname{Sin}^2 A+\operatorname{Sin}^2 B+\operatorname{Sin}^2 C=1-2 \operatorname{Sin} A \operatorname{Sin} B \operatorname{Sin} C$

10. $\text{Cos}^2 A+\text{Cos}^2 B+\text{Cos}^2 C=1-2 \text{ Cos } A \text{ Cos } B \text{ Cos } C$
And hence show that

$$\text{Cos } A.\ \text{Cos } B.\ \text{Cos } C \text{ is less than } \tfrac{1}{2}$$

(K. U. Pre. 1962)

[**Hint** :– (*ii*) L. H. S. is necessarily+ve. So must be the R. H. S. Hence Cos A Cos B Cos C is less than $\frac{1}{8}$]

11. $\text{Sin } \frac{A}{2}+\text{Sin}^2 \frac{B}{2}+\text{Sin}^2 \frac{C}{2}=1-2 \text{ Sin } \frac{A}{2} \text{ Cos } \frac{B}{2} \text{ Sin}^2 \frac{C}{2}$

12. $\text{Sin}^2 \frac{A}{2}+\text{Sin}^2 \frac{B}{2}-\text{Sin}^2 \frac{C}{2}=1-2 \text{ Cos } \frac{A}{2} \text{ Cos } \frac{B}{2} \text{ Sin } \frac{C}{2}$

13. $\text{Cos}^2 \frac{A}{2}+\text{Cos}^2 \frac{B}{2}+\text{Cos}^2 \frac{C}{2}=2\left(1+\text{Sin } \frac{A}{2} \text{ Sin } \frac{B}{2} \text{ Sin } \frac{C}{2}\right)$

14. $\text{Cos}^2 A+\text{Cos}^2 B-\text{Cos}^2 C$
$=1-\text{Sin } A \text{ Sin } B \text{ Cos } C$

15. $\text{Sin}^2 B+\text{Sin}^2 C-\text{Sin}^2 A=2 \text{ Sin } B \text{ Sin } C \text{ Cos } A$

(*D*) Identities holding between *tangents* and *Cotangents* of three angles.

If $A+B+C=180°$, prove that

16. $\tan A+\tan B+\tan C=\tan A \tan B \tan C$

17. $\text{Cot } B.\ \text{Cot } C+\text{Cot } C.\ \text{Cot } A+\text{Cot } A.\ \text{Cot } B=1$

(K. U. Pre. 1962)

18. $\text{Cot } \frac{A}{2}+\text{Cot } \frac{B}{2}+\text{Cot } \frac{C}{2}=\text{Cot } \frac{A}{2} \text{ Cot } \frac{B}{2} \text{ Cot } \frac{C}{2}$

(K. U. 1951)

19. $\tan\dfrac{A}{2}\tan\dfrac{B}{2}+\tan\dfrac{B}{2}\tan\dfrac{C}{2}+\tan\dfrac{C}{2}\tan\dfrac{A}{2}=1$

(*K. U. 1949*)

20. $\text{Cot A}+\text{Cot B}+\text{Cot C}$
$=\text{Cot A. Cot B. Cot C}+\text{Cosec A Cosec B}\times\text{Cosec C}$

(*E*) *Miscellaneous Identities*

If $A+B+C=180°$, prove that

21. $$\frac{\text{Sin }2A+\text{Sin }2B+\text{Sin }2C}{\text{Sin A}+\text{Sin B}+\text{Sin C}}=8\text{ Sin}\frac{A}{2}\text{ Sin}\frac{B}{2}\text{ Sin}\frac{C}{2}$$

(*K. U. 1958*)

22. $$\frac{\text{Cos A}}{\text{Sin B. Sin C}}+\frac{\text{Cos B}}{\text{Sin C. Sin A}}+\frac{\text{Cos C}}{\text{Cos A. Sin B}}=2$$

23. If $x+y+z=xyz$, prove that

(*i*) $$\frac{2x}{1-x^2}+\frac{2y}{1-y^2}+\frac{2z}{1-z^2}=\frac{2x}{1-x^2}\cdot\frac{2y}{1-y^2}\cdot\frac{2z}{1-z^2}$$

(*ii*) $$x(1-y^2)(1-z^2)+y(1-z^2)(1-x^2)+z(1-x^2)(1-y^2)=4xyz$$

[**Hint** :—Result (*ii*) follows immediately by multiplying (*i*) by $(1-x^2)(1-y^2)(1-z^2)$]

If $A+B+C=180°$, prove that :—

24. $$\text{Cos}\frac{A}{2}+\text{Cos}\frac{B}{2}+\text{Cos}\frac{C}{2}=4\text{ Cos}\frac{\pi+A}{4}\text{ Cos}\frac{\pi+B}{4}\text{ Cos}\frac{\pi-C}{4}$$

25. $$\text{Sin}\frac{A}{2}+\text{Sin}\frac{B}{2}+\text{Sin}\frac{C}{2}=1+4\text{ Sin}\frac{\pi-A}{4}\text{ Sin}\frac{\pi-B}{4}\text{ Sin}\frac{\pi-C}{4}$$

6. 6. *Eliminations*

No hard and fast rules can be laid down for eliminating Trigonometrical functions from given equations. However, if, for instance, from two given equations, we get $\text{Sin }\theta = x$ and $\text{Cos }\theta = y$, then we have $1 = \text{Sin}^2\theta + \text{Cos}^2\theta = x^2 + y^2$.

Hence $x^2 + y^2 = 1$ is the required eliminant as it is free from θ. The following solved examples will illustrate the methods.

Ex. 1. Eliminate θ from :—

$$a \text{ Cos } \theta + b \text{ Sin } \theta = C$$
$$b \text{ Cos } \theta - a \text{ Sin } \theta = d$$

Sol. Squaring and adding these two equations, we get

$$a^2(\text{Cos}^2\theta + \text{Sin}^2\theta) + b^2(\text{Cos}^2\theta + \text{Sin}^2\theta) = C^2 + d^2$$

or $\quad a^2 + b^2 = c^2 + d^2$

$$(\because \text{Cos}^2\theta + \text{Sin}^2\theta = 1)$$

which is the eleminant.

Ex. 2. Eliminate θ from the equations :—

$$x \text{ Cos } \theta + y \text{ Sin } \theta = c$$
$$a \text{ Cos } \theta + b \text{ Sin } \theta = d$$

Sol. We have
$$x \text{ Cos } \theta + y \text{ Sin } \theta - c = 0$$
$$a \text{ Cos } \theta + b \text{ Sin } \theta - d = 0$$

By Cross-multiplication, we get

$$\frac{\text{Cos } \theta}{bc - yd} = \frac{\text{Sin } \theta}{dx - ca} = \frac{1}{bx - ay}$$

$$\because \text{ Cos } \theta = \frac{bc - yd}{bx - ay} \text{ and Sin } \theta = \frac{dx - ca}{bx - ay}$$

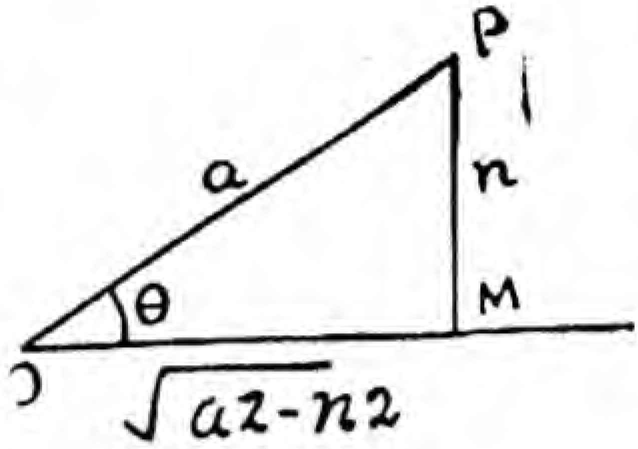
Squaring and adding these two, we get

$$1 \equiv \text{Cos}^2\theta + \text{Sin}^2\theta = \left(\frac{bc - yd}{bx - ay}\right)^2 + \left(\frac{dx - ca}{bx - ay}\right)^2$$

or $(bx - ay)^2 = (bc - yd)^2 + (dx - ca)^2$ which is the required eliminant.

Ex. 3. Eliminate θ from the equations :—

$$x = a \text{ Sin } \theta \text{ ; } y = b \text{ Cot } \theta$$



Sol. From the first equation,

Sin $\theta = \frac{x}{a}$ $\therefore$ cot $\theta = \sqrt{\frac{a^2-x^2}{x}}$

Substituting this value of cot θ in the *second* equation, we get.

$$y = b.\sqrt{\frac{a^2-x^2}{x}}$$

or $\qquad x^2y^2 = b^2(a^2-x^2)$

which is the required eliminant.

Ex. 4. Eliminate θ from :—

Cosec $\theta -$ Sin $\theta = m$ and sec $\theta -$ Cos $\theta = n$

Sol. From the given equations, we get

$1 - \text{Sin}^2\,\theta = m\,\text{Sin}\,\theta$ $\quad$ or $\quad$ $\text{Cos}^2\,\theta = m\,\text{Sin}\,\theta$

and $1 - \text{Cos}^2\,\theta = n\,\text{Cos}\,\theta$ $\quad$ or $\quad$ $\text{Sin}^2\,\theta = n\,\text{Cos}\,\theta$

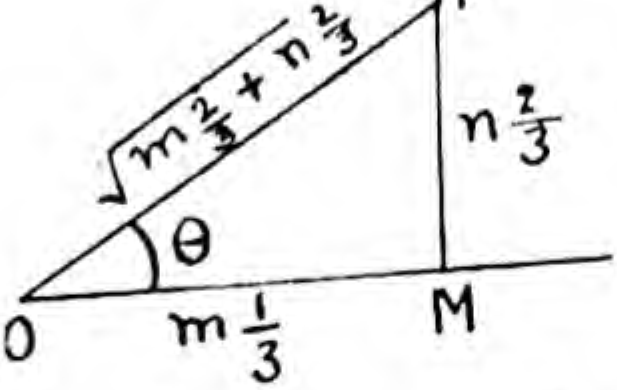
Now $\text{Cos}^2\,\theta = m\,\text{Sin}\,\theta$ $\qquad$ (*i*)

and $\text{Sin}^2\,\theta = n\,\text{Cos}\,\theta$ $\qquad$ (*ii*)

$\therefore$ By dividing (*ii*) by (*i*), we get

$$\tan^2\theta = \frac{n}{m}\,\text{Cot}\,\theta$$

or $\qquad \tan^3\theta = \frac{n}{m}$ $\qquad$ or $\tan\theta = \left(\frac{n}{m}\right)^{\frac{1}{3}}$



$$\therefore\ \text{Sin}\,\theta = \frac{n^{\frac{1}{3}}}{\sqrt{m^{\frac{2}{3}}+n^{\frac{2}{3}}}}$$

$$\text{and Cosec}\,\theta = \frac{\sqrt{m^{\frac{2}{3}}+n^{\frac{2}{3}}}}{n^{\frac{1}{3}}}$$

Substituting these values of Sin θ and Cosec θ in the equation Cosec $\theta -$ Sin $\theta = m$, we have

$$\frac{\sqrt{m^{\frac{2}{3}}+n^{\frac{2}{3}}}}{n^{\frac{1}{3}}}-\frac{n^{\frac{1}{3}}}{\sqrt{m^{\frac{2}{3}}+n^{\frac{2}{3}}}}=m$$

or $$\frac{m^{\frac{2}{3}}}{n^{\frac{1}{3}}\sqrt{m^{\frac{2}{3}}+n^{\frac{2}{3}}}}=m$$

or $$\frac{m^{\frac{2}{3}}}{\sqrt{m^{\frac{2}{3}}+n^{\frac{2}{3}}}}=mn^{\frac{1}{3}}$$

or $$\frac{1}{\sqrt{m^{\frac{2}{3}}+n^{\frac{2}{3}}}}=m^{\frac{1}{3}}n^{\frac{1}{3}}$$

or $m^{\frac{2}{3}}+n^{\frac{2}{3}}=(mn)^{-\frac{2}{3}}$

Which is the required eliminant.

EXERCISE IX

Eliminate θ from the equations :—

1. (*i*) $x=a \text{ Cos } \theta, y=a \text{ Sin } \theta$
   (*ii*) $x=a \text{ Cos } \theta, y=b \text{ Sin } \theta$
   (*iii*) $x=a \text{ Sec } \theta, y=b \text{ Cosec } \theta$
   (*iv*) $x=a \text{ Sec } \theta, y=b \tan \theta$
   (*v*) $x=a \text{ Sec } \theta, y=b \text{ Cot } \theta$
2. (*i*) $x=\text{Sin } \theta+\text{Cos } \theta, y=\text{Sin } \theta-\text{Cos } \theta$
   (*ii*) $x=5 \text{ Cos } \theta-7 \text{ Sin } \theta, y=4 \text{ Cos } \theta+9 \text{ Sin } \theta$
   (*iii*) $3 \tan \theta+\text{Sec } \theta=p, \tan \theta-\text{Sec } \theta=q$
3. $x=\text{Sin } (\theta+\alpha), y=\text{Cos } (\theta-\beta)$
4. $\frac{ax}{\text{Cos } \theta}-\frac{by}{\text{Sin } \theta}=a^2-b^2$ and

   $\frac{ax \text{ Sin } \theta}{\text{Cos}^2 \theta}+\frac{by \text{ Cos } \theta}{\text{Sin}^2 \theta}=0$

5. $x=\text{Cos}^2\,\theta-\text{Sin}^2\,\theta, y=2\,\text{Sin}\,\theta\,\text{Cos}\,\theta$

6. $x=a\,\text{Cos}\,2\theta, y=b\,\text{Sin}\,\theta$

7. $x=\text{Sin}\,\theta+\text{Cos}\,\theta, y=\text{Sin}^3\,\theta+\text{Cos}^3\,\theta$

8. If $\tan\theta+\text{Sin}\,\theta=m$
and $\tan\theta-\text{Sin}\,\theta=n$
prove that $m^2-n^2=4\sqrt{mn}$

9. Eliminate θ and φ from
$\text{Sin}\theta+\text{Sin}\varphi=p$; $\text{Cos}\,\theta+\text{Cos}\,\varphi=q$
and $\text{Cos}\,(\theta-\varphi)=r$

10. If $x=\gamma\,\text{Sin}\,\theta\,\text{Cos}\varphi$
$y=\gamma\,\text{Sin}\,\theta\,\text{Sin}\,\varphi$
$z=\gamma\,\text{Cos}\,\theta$
Show that $x^2+y^2+z^2=\gamma^2$

CHAPTER VII

Trigonometrical Equations

7.1 (*a*) *Values of Sine.*

We know that : —

$\text{Sin } \theta = 0$

$\text{Sin } \pi = 0$, $\text{Sin } 3\pi = 0$

$\text{Sin } 4\pi = 0$, and so on.

$\therefore$ If $\text{Sin } \theta = 0$, then $\theta = 0, \pi, 2\pi, 3\pi, \ldots\ldots$

or

$\theta = n\pi$ where $n = 0$ or anyother $+ve$ or $-ve$ integer.

Hence if *Sin* $\theta = 0$, then

$$\theta = n\pi \text{ where } n = 0, 1, 2, 3.$$

(*b*) *values of Cosine :*

We know that :—

$\text{Cos } \frac{\pi}{2} = 0$, $\text{Cos } \frac{3\pi}{2} = 0$, $\text{Cos } \frac{5\pi}{2} = 0$, $\text{Cos } \frac{7\pi}{2} = 0$ and so on

$\therefore$ If $\text{Cos } \theta = 0$ then

$$\theta = \frac{\pi}{2}; \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots\ldots$$

or $\theta =$ any odd multiple of $\frac{\pi}{2}$

Hence if *Cos* $\theta = 0$, then

$$\theta = (2n+1)\frac{\pi}{2} \text{ where } n = 0, 1, 2, 3, \ldots\ldots$$

7·2. *Find a general expression for angles having the same sine.*

Sol. Let α be the least angle, positive or negative, having the same sine as Sine θ. Then $\text{Sin}\,\theta=\text{Sin}\,\alpha$

or $\text{Sin}\,\theta-\text{Sin}\,\alpha=0$

or $2\,\text{Cos}\frac{\theta+\alpha}{2}\,\text{Sin}\,\frac{\theta-\alpha}{2}=0$

Either $\text{Cos}\,\frac{\theta+\alpha}{2}=0$

$\therefore\ \frac{\theta+\alpha}{2}=(2\gamma+1)\frac{\pi}{2}$ [article 7 (*b*)]

or $\theta=(2\gamma+1)\pi-\alpha$......(1)

or $\text{Sin}\,\frac{\theta-\alpha}{2}=0$

$\therefore\ \frac{\theta-\alpha}{2}=p\pi$ (article 7 (*a*))

or $\theta=2p\pi+\alpha$.........(2)

Combining results (1) and (2) we get

$\theta=n\pi+(-1)^n\alpha$ *where n is zero or a −ve or +ve integer. This combined result agrees with result (1) if n is odd and with result (2) if n is even.*

Cor. *find a general expression for all angles having the same Cosecant.*

Here $\text{Cosec}\,\theta=\text{Cosec}\,\alpha$

Which gives $\text{Sin}\,\theta=\text{Sin}\,\alpha$

(This is the same as article 7·2)

7·3. *Find a general expression for all angles having the same Cosine.*

Sol. Let α be the *least* angle having the same *Cosine* as $\text{Cos}\,\theta$.

i.e. $\text{Cos}\,\theta=\text{Cos}\,\alpha$

or $\text{Cos}\,\alpha-\text{Cos}\,\theta=0$

$\therefore\ 2\,\text{Sin}\,\frac{\theta+\alpha}{2}\,\text{Sin}\,\frac{\theta-\alpha}{2}=0$

Either $\text{Sin}\,\frac{\theta+\alpha}{2}=0$

Which gives $\frac{\theta+\alpha}{2}=\text{K}\pi$

or $\theta=2\text{K}\pi-\alpha$

Or $\text{Sin}\frac{\theta-\alpha}{2}=0$

Which gives $\frac{\theta-\alpha}{2}=p\pi$

or $\theta=2p\pi+\alpha$

Combining these two results, we get

$$\theta = 2n\pi \pm \alpha \text{ where } n \text{ is zero}$$

or a *positive or negative* integer.

Cor. *Find the general expression for all angles having the same Secant.*

Sol. Let α be the least angle (+*ve* or −*ve*) having same Secant as Sec θ, then

$$\text{Sec } \theta = \text{Sec } \alpha$$

Which gives Cos α = Cos θ

(This is the same as article 7.3)

7.4 *Find the general expression for all angles having the same tangent.*

Sol. Let α be the least angle having the same tangent and tangent θ

i.e.
$$\tan \theta = \tan \alpha$$

$$\frac{\text{Sin } \theta}{\text{Cos } \theta} = \frac{\text{Sin } \alpha}{\text{Cos } \theta}$$

$$\frac{\text{Sin } \theta \text{ Cos } \alpha - \text{Cos } \theta \text{ Sin } \alpha}{\text{Cos } \theta \text{ Cos } \alpha} = 0$$

or
$$\frac{\text{Sin } (\theta - \alpha)}{\text{Cos } \theta \text{ Cos } \alpha} = 0$$

$\therefore$
$$\text{Sin } (\theta - \alpha) = 0$$

Which gives $\theta - \alpha = n\pi$ [article 7 (*a*)]

$$\therefore \quad \theta = n\pi + \alpha$$

Cor. *Find the general expression for all angles having the same cotangent.*

Sol. Let α be the least angle (+*ve* or −*ve*) having the Same Cotangent as Cot θ, then.

$$\text{Cot } \theta = \text{Cot } \alpha$$

which gives tan θ = tan α

(This is the same as article 7·4)

Note : *In all the cases, we have to find α the least angle, and put it in radians.*

Solved Examples

Ex. 1. Solve the following :—

(*i*) $\sin\theta = \frac{\sqrt{3}}{2}$ (*ii*) $\sec\theta = \frac{2}{\sqrt{3}}$ (*iii*) $\tan\theta = \sqrt{3}$

Sol.

(*i*) $\sin\theta = \frac{\sqrt{3}}{2}$ Here the *least* angle for which

$\sin\theta = \frac{\sqrt{3}}{2}$ is 60°

$\therefore \quad \alpha = 60^\circ = \frac{\pi}{3}$

Hence $\theta = n\pi + (-1)^n \frac{\pi}{3}$ (article 7·2)

(*ii*) $\sec\theta = \frac{2}{\sqrt{3}}$

$\therefore \cos\theta = \frac{\sqrt{3}}{2}$ Here $\alpha = 30^\circ = \frac{\pi}{6}$

Hence $\theta = 2n\pi \pm \frac{\pi}{6}$ (article 7·3)

(*iii*) $\tan\theta = \sqrt{3}$ Here $\alpha = 60^\circ = \frac{\pi}{3}$

Hence $\theta = n\pi + \frac{\pi}{3}$ (article 7·4)

Ex. 2 Solve the following equations :—

(*i*) $\cos\theta = -\frac{1}{2}$ (*ii*) $\tan\theta + 1 = 0$ (*iii*) $2\sin\theta + 1 = 0$

Sol. (*i*) $\cos\theta = -\frac{1}{2}$

The least angle lying between 0 and 2π and satisfying this equation is 120° or $\frac{2\pi}{3}$

$\therefore \quad \cos\theta = \cos\frac{2\pi}{3}$

Hence $\theta=2n\pi\pm\frac{2\pi}{3}$

(*ii*) Here $\tan\theta=-1$

The least angle lying between 0 and 2π and satisfying the given equation 135° or $=\frac{3\pi}{4}$

$\therefore \quad \theta=n\pi+\frac{3\pi}{4}$

(*iii*) Here $\sin\theta=-\frac{1}{2}$ and the least angle lying between 0 and 2π and satisfying this equation is 210° or $\frac{7\pi}{6}$

$\therefore \quad \theta=n\pi+(7)^n\frac{7\pi}{6}$

Ex. 3. Solve :—

(*i*) $\cos 9\theta=\frac{1}{\sqrt{2}}$ (*ii*) $\sin 2\theta=2\cos\theta$

(*iii*) $4\sin^2\theta=3$

Sol. (*i*) Here $\cos 9\theta=\frac{1}{\sqrt{2}}=\cos\frac{\pi}{4}$

$\therefore \quad 9\theta=2n\pi\pm\frac{\pi}{4}$

which gives $\theta=\frac{1}{9}\left[2n\pi\pm\frac{\pi}{4}\right]$

(*ii*) Here $\sin 2\theta-2\cos\theta=0$

or $2\sin\theta\cos\theta-2\cos\theta=0$

or $2\cos\theta(\sin\theta-1)=0$

either $2\cos\theta=0$ or $\sin\theta-1=0$

$\therefore \quad \cos\theta=0$

$\therefore \qquad \theta=(2n+1)\dfrac{\pi}{2} \therefore \ \text{Sin}\,\theta=1=\text{Sin}\dfrac{\pi}{2}$

$$\therefore \quad \theta=p\pi+(-1)^p\frac{\pi}{2}$$

(*iii*) $4\,\text{Sin}^2\theta=3$

This gives $\text{Sin}\ \theta=\pm\dfrac{\sqrt{3}}{2}$

Now $\qquad \text{Sin}\,\theta=\dfrac{\sqrt{3}}{2}=\text{Sin}\dfrac{\pi}{3}$

$\therefore \qquad \theta=n\pi+(-1)^n\dfrac{\pi}{3}$

Again, $\qquad \text{Sin}\,\theta=-\dfrac{\sqrt{3}}{2}=\text{Sin}\left(\pi+\dfrac{\pi}{3}\right)=\text{Sin}\dfrac{4\pi}{3}$

$\therefore \qquad \theta=p\pi+(-1)^p\dfrac{4\pi}{3}$

or $\qquad \theta=p\pi-(-1)^p\dfrac{\pi}{3}$ if we take $-ve$ angle.

Ex. 4. What is the most general value of θ which satisfies the equations :—

$$\text{Sec}\,\theta=-\sqrt{2}\ ;\ \text{Cot}\,\theta=1$$

Sol. $\text{Sec}\ \theta=-\sqrt{2}$

or $\text{Cos}\,\theta=-\dfrac{1}{\sqrt{2}}=\text{Cos}\dfrac{3\pi}{4}$

$\therefore \qquad \theta=2n\pi\pm\dfrac{3\pi}{4}$ ......(1)

Also, $\text{Cot}\ \theta=1$

or $\qquad \tan\ \theta=1=\tan\ \dfrac{\pi}{4}$

$\therefore \qquad \theta=n\pi+\dfrac{\pi}{4}$ ......(2)

Clearly the values common to (1) and (2) are given by

$$\theta=(2m+1)\pi+\frac{\pi}{4}$$

Ex. 5. Solve the equations :—

(*i*) $\cos 3\theta=\sin 2\theta$ (*ii*) $\tan n\theta=\cot m\theta$

(*iii*) $4\cos^2\theta-4\sin\theta=1$

Sol. (*i*) $\cos 3\theta=\sin 2\theta=\cos\left(\frac{\pi}{2}-2\theta\right)$

or $3\theta=2n\pi\pm\left(\frac{\pi}{2}-2\theta\right)$ $\left[\because \alpha=\frac{\pi}{2}-2\theta\right]$

or $3\theta\pm2\theta=2n\pi\pm\frac{\pi}{2}$

or $(3\pm2)\theta=(4n\pm1)\frac{\pi}{2}$

$$\therefore \theta=\frac{(4n\pm1)\pi}{2(3\pm2)}$$

(*ii*) $\tan n\theta=\cot m\theta=\tan\left(\frac{\pi}{2}-m\theta\right)$

or $n\theta=K\pi+\left(\frac{\pi}{2}-m\theta\right)$ $\left[\because \alpha=\frac{\pi}{2}-m\theta\right]$

or $(n+m)\theta=k\pi+\frac{\pi}{2}$

or $(m+n)\theta=\frac{2k\pi+\pi}{2}$

$$=(2k+1)\frac{\pi}{2}$$

$$\therefore \theta=\frac{2k+1}{m+n}\cdot\frac{\pi}{2}$$

(*iii*) $4\cos^2\theta-4\sin\theta=1$

or $4(1-\sin^2\theta)-4\sin\theta-1=0$

(*Please note this Step.*)

or $4 \sin^2 \theta + 4 \sin \theta - 3 = 0$

or $\sin \theta = \dfrac{-4 \pm \sqrt{16+48}}{8} = \dfrac{-4 \pm 8}{8} = \frac{1}{2}, -\frac{3}{2}$

Now $\sin \theta = \frac{1}{2} = \sin \dfrac{\pi}{6}$ Here $\alpha = \dfrac{\pi}{6}$

$\therefore \theta = n\pi + (-1)^n \dfrac{\pi}{6}$

The second value being $-\frac{3}{2}$ is impossible because $\sin \theta$ cannot be numerically greater than 1.

EXERCISE X

Find the most general values of θ satisfying the equations :—

1. $\sin 2\theta = 0$
2. $\sin \theta = \frac{1}{2}$
3. $\sin \theta = -\frac{1}{2}$
4. $\sin 3\theta = \dfrac{\sqrt{3}}{2}$
5. $\sin \theta = p$
6. $\operatorname{cosec} \theta = \dfrac{1}{q}$
7. $\sin 2\theta = \sin 2\alpha$
8. $\operatorname{cosec} \theta = \operatorname{cosec} \alpha$
9. $\cos \theta = \dfrac{1}{\sqrt{2}}$
10. $\cos \theta = -\frac{1}{2}$
11. $\cos 4\theta = \dfrac{\sqrt{3}}{2}$
12. $\cos \theta = p$
13. $\cos m\theta = \cos n\theta$
14. $\cot \theta = \sqrt{3}$
15. $\tan \theta = -1$
16. $\tan \theta = \frac{3}{4}$
17. $\tan \theta = p$
18. $\cos 3x = \sin 2x$
19. $\cos m\theta = \sin n\theta$
20. $\tan 2\theta = \cot 5\theta$
21. $\cot^2 \theta = 3$
22. $\tan 3\theta \tan 5\theta = 1$

(**Hint** :—$\tan 3\theta = \cot 5\theta$)

23. $5 \tan^4 \theta - 1 = 4 \tan^2 \theta$
24. $\sin^2 \theta - 2 \cos \theta + \frac{1}{4} = 0$
25. $2 \cos^2 \theta - 7 \cos \theta + 5 = 0$
26. $\sec^4 \theta - 6 \sec^2 \theta + 8 = 0$

Find the most general value of θ satisfying the following equations simultaneously :—

27. $\sin \theta = -\frac{1}{2}$ and $\tan \theta = \dfrac{1}{\sqrt{3}}$

28. Cot $\theta = -\sqrt{3}$ and Sin $\theta = -\frac{1}{2}$

29. Sec $\theta = -\sqrt{2}$ and Cot $\theta = 1$

Solve the equations :—

30. Cos $(A-B) = \frac{1}{2}$ and Sin $(A+B) = \frac{1}{2}$

31. Cos $(2x+3y) = \frac{1}{2}$ and Cos $(3x+2) = \frac{\sqrt{3}}{2}$

(*P. U. 1944*)

32. $\tan(A+B+C) = \sqrt{3}$

$\tan(A-B+C) = 1$

and $\tan(A+B-C) = \frac{1}{\sqrt{3}}$

7.5 *Solution of different types of Trigonometrical Equations.*

(*a*) *Equations of the form a* Cos $\theta + b$ Sin $\theta = c$

In this type of equation, we proceed as under :—

Put $a = \gamma$ *Cos* φ and $b = \gamma$ Sin φ where γ is a +ve quantity

Squaring and adding, we get

$\sqrt{a^2+b^2} = \gamma$ and by division, we get $\tan\varphi = \frac{b}{a}$ After making these substitutions, the given equation is reduced to the following form γ

$$\text{Cos}\,\varphi\,\text{Cos}\,\theta + \gamma\,\text{Sin}\,\varphi\,\text{Sin}\,0 = C$$

$$\text{or } \gamma\,\text{Cos}\,(\theta-\varphi) = C$$

$$\text{or Cos}\,(0-\varphi) = \frac{C}{\gamma} = \frac{C}{\sqrt{a^2+c^2}} = \text{Cos}\,\alpha \text{ (Say)}$$

$\therefore$ By article 7.3, we have $\theta - \varphi = 2n\pi \pm \alpha$

Which gives $0 = 2n\pi \pm \alpha + \varphi$

Where $\tan\varphi = \frac{b}{a}$ and Cos $\alpha = \frac{C}{\sqrt{a^2+b^2}}$

Note :—The equation a Cos $\theta + b$ Sin $\theta = c$ can also be solved by the substitution $a = \gamma$ Sin φ and $b = \gamma$ Cos φ and we get two different solutions by adopting two different methods. But these do differ from each other at all, which is shown below :—

Ex. Solve the equation $\sqrt{3}\,\text{Cos}\,\theta+\text{Sin}\theta=\sqrt{2}$

Sol. First method :—

Put $\sqrt{3}=\gamma\ \text{Cos}\ \alpha$ and $1=\gamma\ \text{Sin}\ \alpha$ so that $\gamma=2$ and $\tan\alpha=\dfrac{1}{\sqrt{3}}$ or $\alpha=\dfrac{\pi}{6}$ Now the given equation is reduced to $\gamma\ \text{Cos}\ (\theta-\alpha)=\sqrt{2}$

$$\text{or Cos}\ (\theta-\alpha)=\frac{\sqrt{2}}{\gamma}=\frac{\sqrt{2}}{2}=\frac{\sqrt{1}}{2}=\text{Cos}\ \frac{\pi}{4}$$

$$\text{or}\qquad \theta-\alpha=2n\pi\pm\frac{\pi}{4}\qquad\qquad \text{(article 7·3)}$$

$$\text{or}\qquad \theta=2\ n\pi\pm\frac{\pi}{4}+\alpha$$

$$=2\ n\pi\pm\frac{\pi}{4}+\frac{\pi}{6}\ldots(i)\ \left(\because\alpha=\frac{\pi}{6}\right)$$

Second Method :—

Put $\sqrt{3}=\gamma\ \text{Sin}\ \alpha$ and $1=\gamma\ \text{Cos}\ \alpha$

So that $2=\gamma$ and $\tan\ \alpha=\sqrt{3}$

$$\text{or}\qquad \alpha=\frac{\pi}{3}$$

Now the equation is reduced to

$\gamma\ \text{Sin}\ (\theta+\alpha)=\sqrt{2}$

$$\text{or Sin}\ (\theta+\alpha)=\frac{\sqrt{2}}{\gamma}=\frac{\sqrt{2}}{2}=\frac{1}{\sqrt{2}}=\text{Sin}\ \frac{\pi}{4}$$

$$\text{or}\qquad \theta+\alpha=n\pi+(-1)^n\ \frac{\pi}{4}$$

$$\therefore\qquad \theta=n\ \pi+(-1)^n\ \frac{\pi}{4}-\alpha$$

$$=n\pi+(-1)^n\frac{\pi}{4}-\frac{\pi}{3}\qquad\ldots\ldots(2)$$

We shall now show that solutions (1) and (2) are the same.

When n is even, (2) takes the form

$$2k\pi + \frac{\pi}{4} - \frac{\pi}{3} \text{ i.e., } 2k\pi - \frac{\pi}{12}$$

When n is odd, (2) takes the form

$$(2m+1)\pi - \frac{\pi}{4} - \frac{\pi}{3} \text{ i.e., } 2m\pi + \pi - \frac{7\pi}{12}$$

$$\text{or } 2m\pi + \frac{5\pi}{12}$$

Now (2) takes the following two forms :—

(i) $2k\pi - \dfrac{\pi}{12}$ and (ii) $2m\pi + \dfrac{5\pi}{12}$

But $\dfrac{5\pi}{12} = \dfrac{\pi}{4} + \dfrac{\pi}{6}$ and $-\dfrac{\pi}{12} = -\dfrac{\pi}{4} + \dfrac{\pi}{6}$

Hence (2) can be put as

$$2n\pi \pm \frac{\pi}{4} + \frac{\pi}{6} \text{ which is the same as solution (1)}$$

Note (2) *The student is at liberty to adopt either the First Method or the Second Method.*

(*b*) *Equations reducible to the form*

$$a \text{ Cos } \theta + b \text{ Sin } \theta = C$$

Ex. $\sqrt{2} \text{ Sec } \theta + \tan \theta = 1$

Sol. This can be put as

$$\sqrt{2} \cdot \frac{1}{\text{Cos } \theta} + \frac{\text{Sin } \theta}{\text{Cos } \theta} = 1$$

$$\text{or } \sqrt{2} + \text{Sin } \theta = \text{Cos } \theta$$

This gives $\text{Cos } \theta - \text{Sin } \theta = \sqrt{2}$ and can be solved by the method explained above.

(*c*) *Equations involving more than two multiple angles.*

Ex. Solve the equation

$$\text{Cos } x + \text{Cos } 2x + \text{Cos } 3x = 0$$

Sol. The equation can be written as

$$(\text{Cos } 3x+\text{Cos } x)+\text{Cos } 2x=0$$

or $$2\text{ Cos}\frac{3x+x}{2}\text{ Cos }\frac{3x-x}{2}+\text{Cos } 2x=0$$

or $$2\text{ Cos } 2x\text{ Cos } x+\text{Cos } 2x=0$$

or $$\text{Cos } 2x(2\text{ Cos } x+1)=0$$

Either $\text{Cos } 2x=0$
which gives

$$2x=(2n+1)\frac{\pi}{2}$$

or $x=(2n+1)\frac{\pi}{4}$

or $2\text{ Cos } x+1=0$
which gives

$$\text{Cos } x=-\tfrac{1}{2}$$
$$=\text{Cos }\frac{2\pi}{3}$$

or $x=2n\pi\pm\frac{2\pi}{3}$

Solved Examples

Ex. 1. Solve the equation

$$\text{Sin } x-\text{Cos } x=\sqrt{2}$$

(K. U. Pre. 1962)

Sol. $\text{Sin } x-\text{Cos } x=\sqrt{2}$

Put $1=\gamma\text{ Cos }\theta$ and $1=\gamma\text{ Sin }\theta$

So that $\sqrt{2}=\gamma$ and $\tan\theta=1$ or $\theta=\frac{\pi}{4}$ The given equation, therefore, takes the form

$$\gamma(\text{Cos }\theta\text{ Sin } x-\text{Sin }\theta\text{ Cos } x)=\sqrt{2}$$

or $$\sqrt{2}\text{ Sin }(x-\theta)=\sqrt{2}$$

or $$\text{Sin }(x-\theta)=1=\text{Sin}\frac{\pi}{2}$$

or $$x-\theta=n\pi+(-1)^n\frac{\pi}{2}$$

or $$x=n\pi+(-1)^n\frac{\pi}{2}+\theta$$

$$=n\pi+(-1)^n\frac{\pi}{2}+\frac{\pi}{4}$$

Ex. 2. Solve the equation

$$\text{Cosec }\theta=\sqrt{3}+\text{Cot }\theta$$

Sol. The equation can be reduced to the form $\sqrt{3}\text{ Sin }\theta + \text{Cos }\theta=1$

(Multiplying both sides by Sin θ)

Put $\sqrt{3}=r\text{ Cos }\varphi$ and $1=r\text{ Sin }\varphi$

So that $2=r$ and $\tan\varphi=\dfrac{1}{\sqrt{3}}$ or $\varphi=\dfrac{\pi}{6}$

$\therefore$ The equation becomes

$$r\text{ Sin }(\theta+\varphi)=1$$

or $$\text{Sin }(\theta+\varphi)=\frac{1}{r}=\frac{1}{2}=\text{Sin }\frac{\pi}{}$$

or $$\theta+\varphi=n\pi+(-1)^{n}\frac{\pi}{}$$

or $$\theta=n\pi+(-1)^{n}\frac{\pi}{6}-\varphi$$

$$=n\pi+(-1)^{n}\frac{\pi}{6}-\frac{\pi}{6}$$

Ex. 3. If $\tan(\pi\text{ Cos }\theta)=\text{Cot}(\pi\text{ Sin }\theta)$

Prove that :—

$$\text{Cos}\left(\theta-\frac{\pi}{4}\right)=\frac{1}{2\sqrt{2}}$$

Sol. $\tan(\pi\text{ Cos }\theta)=\text{Cot }(\pi\text{ Sin }\theta)$

$$=\tan\left(\frac{\pi}{2}-\pi\text{ Sin }\theta\right)$$

$$\therefore\ \pi\text{ Cos }\theta=\frac{\pi}{2}-\pi\text{ Sin }\theta$$

or $$\text{Cos }\theta+\text{Sin }\theta=\tfrac{1}{2}$$

Putting $1=\gamma\text{ Cos }\alpha$ and $1=\gamma\text{ Sin }\alpha$

we get :—(i) $\sqrt{2}=\gamma$ and (ii) $\alpha=\frac{\pi}{4}$

$\therefore$ the equation takes the form

$$\gamma\,(\text{Cos}\,\theta\,\text{Cos}\,\alpha+\text{Sin}\,\theta\,\text{Sin}\,\alpha)=\tfrac{1}{2}$$

or $\gamma\,\text{Cos}\,(\theta-\alpha)=\frac{1}{2}$

or $\sqrt{2}\,\text{Cos}\left(\theta-\frac{\pi}{4}\right)=\frac{1}{2}$

$\therefore$ $\text{Cos}\left(\theta-\frac{\pi}{4}\right)=\frac{1}{2\sqrt{2}}$

Ex. 4. Solve the Equations

$$\text{Cos}\,5\,\theta-\text{Sin}\,\theta=\text{Sin}\,3\,\theta-\text{Cos}\,3\,\theta$$

Sol. The equation can be put as

$$(\text{Cos}\,5\theta+\text{Cos}\,3\theta)-(\text{Sin}\,3\,\theta+\text{Sin}\,\theta)=0$$

or $$2\,\text{Cos}\frac{5\theta+3\theta}{2}\,\text{Cos}\,\frac{5\theta-3\theta}{2}-2\,\text{Sin}\,\frac{3\theta+\theta}{2}\,\text{Cos}\,\frac{3\theta-\theta}{2}=0$$

or $2\,\text{Cos}\,4\theta\,\text{Cos}\,\theta-2\,\text{Sin}\,2\theta\,\,\text{Cos}\,\theta=0$

or $2\,\text{Cos}\,\theta\,(\text{Cos}\,4\theta-\text{Sin}\,2\theta)=0$

Either $2\,\text{Cos}\,\theta=0$
which gives $\text{Cos}\,\theta=0$
or $\theta=(2n+1)\frac{\pi}{2}$

or $\text{Cos}\,4\theta-\text{Sin}\,2\theta=0$
or $\text{Cos}\,4\theta=\text{Sin}\,2\theta$

$$=\text{Cos}\left(\frac{\pi}{2}-2\theta\right)$$

or $4\theta=n\pi\pm\left(\frac{\pi}{2}-2\theta\right)$

or $4\theta\pm2\theta=n\pi\pm\frac{\pi}{2}$

or $(4\pm2)\,\theta=(2n\pm1)\,\frac{\pi}{2}$

$\therefore$ $\theta=(2n+1)\,\frac{\pi}{12}$ and $\theta=(2n\quad 1)\frac{\pi}{4}$

EXERCISE XI

Solve the following equations :—

1. $\sin\theta+\cos\theta=\sqrt{2}$
2. $\cos\theta+\sqrt{3}\sin\theta=\sqrt{2}$
3. $\cos\theta+\sqrt{3}\sin\theta=2$
4. $\sin\theta+\sqrt{3}\cos\theta=1$
5. $\sqrt{3}\sin\theta-\cos\theta=\sqrt{2}$
6. $\operatorname{cosec}\theta=\cot\theta+\sqrt{3}$
7. $\sqrt{3}\cot\theta=2\operatorname{cosec}\theta-1$
8. $\sqrt{3}\tan\theta=1+\sec\theta$
9. $\cos 3\theta+\cos 5\theta=\cos\theta$
10. $\sin 3\theta+\sin\theta=\sin 2\theta$
11. $\sin 7\theta-\sin 3\theta=\sin\theta$
12. $\sin\theta+\sin 3\theta-\sin 4\theta=0$
13. $\sin 2x+\sin 4x=\cos x+\cos 3x$
14. $\cos 2\theta-5\cos\theta=2$
15. $\cos m\theta=\cos n\theta$
16. $\cos m\theta=\sin n\theta$
17. $\tan(\pi\cot\theta)=\cot(\pi\tan\theta)$
18. $\sin m\theta=\cos n\theta$
19. $\cos 3x=\sin x$
20. $\tan 3x=\cot x$
21. $2(\sin^4\theta+\cos^4\theta)=1$
22. $\cos 3\theta+8\cos^3\theta=0$
23. $\sin 3\theta=8\sin^2\theta$
24. $\cos^2\theta-\cos\theta\sin\theta-\sin^2\theta=1$
25. $\tan\theta+\tan 2\theta+\tan 3\theta=0$

CHAPTER VIII

Relations between the sides and the angles of a triangle.

8.1. In the present chapter, we will establish certain important relations between the sides and the angles of a triangle. The student will thus come across many important things like "Sine Formula", "Cosine Formula", "Projection Formula", etc. For the sake of convenience, we denote the angles of the triangle ABC by the capital letters A, B, and C, and the sides opposite to these angles by small letters a, b and c respectively.

8.2. *Sine Formula*

To prove that in any triangle ABC

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$$

i.e., the sines of the angles of a triangle are proportional the opposite sides.

Proof :—In the $\triangle$ABC

draw AD $\perp$ BC or BC produced

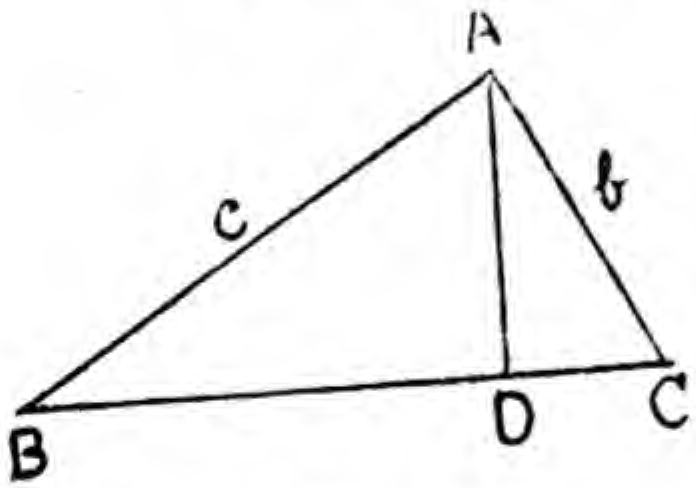


Fig. (i)

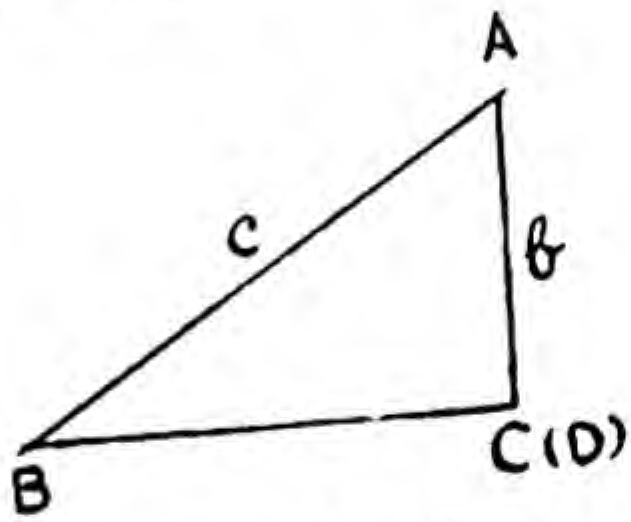


Fig. (ii)

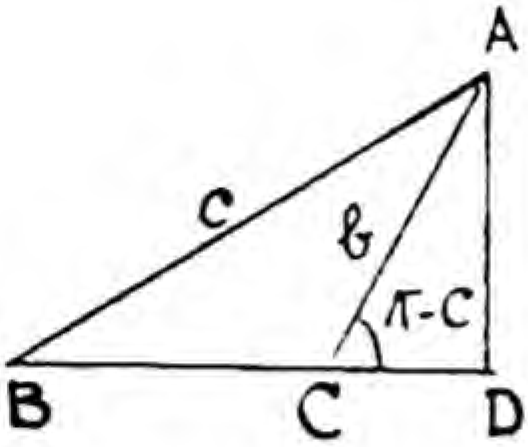


Fig. (iii)

Case I. When ABC is an acute angled triangle. In the rt. $\angle$d $\triangle$ABD,

$\frac{AD}{c} = \text{Sin B}$

$\therefore AD = c \text{ Sin B} \ldots (i)$

Also, in the rt. $\angle$d $\triangle$ ADC,

$\frac{AD}{b} = \text{Sin C}$

$\therefore AD = b \text{ Sin C} \ldots \ldots (ii)$

from (i) and (ii), we have c Sin B $= b$ Sin C

$\therefore \frac{c}{\text{Sin C}} = \frac{b}{\text{Sin B}}$

Similarly,

$\frac{a}{\text{Sin A}} = \frac{b}{\text{Sin B}}$

Hence $\frac{a}{\text{Sin A}} = \frac{b}{\text{Sin B}} = \frac{c}{\text{Sin C}}$

Case II. When ABC is a rt. $\angle$d $\triangle$. In the rt. $\angle$d $\triangle$ ABC, $\frac{AD}{C} = \text{Sin B}$

$\therefore AD = c \text{ Sin B} \ldots (i)$

Also $AD = AC$
$= AC \text{ Sin C}$
$(\because \text{Sin C} = \text{Sin } 90 = 1)$

$\therefore AD = b \text{ Sin C} \ldots (ii)$

from (i) and (ii) we have

$b \text{ Sin C} = c \text{ Sin B}$

$\therefore \frac{b}{\text{Sin B}} = \frac{c}{\text{Sin C}}$

Similarly

$\frac{a}{\text{Sin A}} = \frac{b}{\text{Sin B}}$

Hence $\frac{a}{\text{Sin A}} = \frac{b}{\text{Sin}} = \frac{c}{\text{Sin C}}$

Case III. When ABC is an obtuse $\angle$d $\triangle$. In the rt. $\angle$d $\triangle$ABD, $\frac{AD}{c} = \text{Sin B}$

$\therefore AD = c \text{ Sin B} \ldots (i)$

Also, from the rt. $\angle$d $\triangle$ACD,

$\frac{AD}{b} = \text{Sin } (\pi - C) = \text{Sin C}$

$\because AD = b \text{ Sin C} \ldots (ii)$

from (i) and (ii) we get b Sin C $= c$ Sin B

$\therefore \frac{b}{\text{Sin B}} = \frac{c}{\text{Sin C}}$

Similarly,

$\frac{a}{\text{Sin A}} = \frac{b}{\text{Sin B}}$

Hence $\frac{a}{\text{Sin A}} = \frac{b}{\text{Sin B}} = \frac{c}{\text{Sin C}}$

Note :—*The student is advised to draw all the three figures and derive the formula from all of them.*

Solved Examples (By means of Sine Formula.)

Ex. 1. Prove that in any $\triangle$ ABC

$$a \text{ Cos} \frac{B-C}{2} = (b+c) \text{ Cos } \frac{B+C}{2}$$

(K. U. Pre. 1962)

Sol. We have to prove that

$$a \text{ Cos} \frac{B-C}{2} = (b+c) \text{ Cos } \frac{B+C}{2}$$

Which is the same as

$$\frac{a}{b+c} = \frac{\text{Cos } \frac{B+C}{2}}{\text{Cos } \frac{B-C}{2}}$$

$$\text{Now } \frac{a}{\text{Sin A}} = \frac{b}{\text{Sin B}} = \frac{c}{\text{Sin C}} = k \text{ (Say)}$$

$$\therefore a = k \text{ Sin A}, b = k \text{ Sin B}, c = k \text{ Sin C}$$

$$\text{L. H. S.} = \frac{a}{b+c} = \frac{k \text{ Sin A}}{k(\text{Sin B} + \text{Sin C})} = \frac{\text{Sin A}}{\text{Sin B} + \text{Sin C}}$$

$$= \frac{2 \text{ Sin} \frac{A}{2} \text{ Cos } \frac{A}{2}}{2 \text{ Sin} \frac{B+C}{2} \text{ Cos } \frac{B-C}{2}}$$

$$= \frac{\text{Cos} \frac{B+C}{2} \text{ Cos } \frac{A}{2}}{\text{Cos } \frac{A}{2} \text{ Cos} \frac{B-C}{2}}$$

$$\left[\begin{array}{l} \because (i) \text{ Sin } \frac{A}{2} = \text{Sin } \left(\frac{\pi}{2} - \frac{B+C}{2} \right) = \text{ Cos } \frac{B+C}{2} \\ (ii) \text{ Sin } \frac{B+C}{2} = \text{Sin} \left(\frac{\pi}{2} - \frac{A}{2} \right) = \text{ Cos } \frac{A}{2} \end{array} \right]$$

$$=\frac{\cos\dfrac{B+C}{2}}{\cos\dfrac{B-C}{2}}=\text{R. H. S.}$$

Note :—*If possible, small letters should be kept on one side before the question is attempted.*

Ex. 2. In any triangle ABC, prove that

(*i*) $\dfrac{\cos B}{\cos C}=\dfrac{c-b\cos A}{b-c\cos A}$

(*ii*) $\dfrac{a\sin(B-C)}{b^2-c^2}=\dfrac{b\sin(C-A)}{c^2-a^2}=\dfrac{c\sin(A-B)}{a^2-b^2}$

(*K. U. 1961*)

Sol. (*i*) $\dfrac{\cos B}{\cos C}=\dfrac{c-b\cos A}{b-c\cos A}$

R. H. S.
$$=\frac{c-b\cos A}{b-c\cos A}=\frac{k\sin C-k\sin B\cos A}{k\sin B-k\sin C\cos A}$$

$$\left(\because\ \begin{matrix} b=k\sin B \text{ and} \\ c=k\sin C\end{matrix}\right)$$

$$=\frac{\sin C-\sin B\cos A}{\sin B-\sin C\cos A}=\frac{\sin(\pi-\overline{A+B})-\sin B\cos A}{\sin(\pi-\overline{C+A})-\sin C\cos A}$$

$$=\frac{\sin(A+B)-\sin B\cos A}{\sin(C+A)-\sin C\cos A}\left[\because\ \sin(\pi-\theta)=\sin\theta\right]$$

$$=\frac{\sin A\cos B+\cos A\sin B-\sin B\cos A}{\sin C\cos A+\cos C\sin A-\sin C\cos A}$$

$$=\frac{\sin A\cos B}{\cos C\sin A}$$

$$=\frac{\cos B}{\cos C}=\text{L. H. S.}$$

(*ii*) $$\frac{a\sin(B-C)}{b^2-c^2}=\frac{k\sin A\sin(B-C)}{k^2\sin^2 B-k^2\sin^2 C}$$

$$=\frac{1}{k}\cdot\frac{\sin(\pi-\overline{B+C})\sin(B-C)}{\sin^2 B-\sin^2 C}$$

$$= \frac{1}{k} \cdot \frac{\text{Sin}(B+C)\,\text{Sin}(B-C)}{\text{Sin}(B+C)\,\text{Sin}(B-C)} = \frac{1}{k}$$

$$[\because \text{Sin}^2 B - \text{Sin}^2 C = \text{Sin}(B+C)\,\text{Sin}(B-C)]$$

Similarly, we can show that

$$\frac{b\,\text{Sin}(C-A)}{c^2-a^2} = \frac{c\,\text{Sin}(A-B)}{a^2-b^2} = \frac{1}{k}$$

$$\therefore \frac{a\,\text{Sin}(B-C)}{b^2-c^2} = \frac{b\,\text{Sin}(C-A)}{c^2-a^2} = \frac{c\,\text{Sin}(A-B)}{a^2-b^2} = \frac{1}{k}$$

Ex. 3. In any triangle ABC, if a Cos A $= b$ Cos B, then the triangle is either isosceles or right-angled.

Sol. $a\,\text{Cos A} = b\,\text{Cos B}$

or $k\,\text{Sin A Cos A} = k\,\text{Sin B Cos B}$ $\left(\because \begin{matrix} a = k\,\text{Sin A and} \\ b = k\,\text{Sin B} \end{matrix}\right)$

or $\text{Sin A Cos A} = \text{Sin B Cos B}$

or $\frac{2\,\text{Sin A Cos A}}{2} = \frac{2\,\text{Sin B Cos B}}{2}$

or $\frac{\text{Sin 2A}}{2} = \frac{\text{Sin 2B}}{2}$

or $\text{Sin 2A} = \text{Sin 2B}$

or $2A = 2B$

or $A = B$

Which shows that the triangle is isosceles.

Again, $\text{Sin 2A} = \text{Sin 2B}$

$= \text{Sin}(\pi - 2B)$ $[\because \text{Sin}(\pi-\theta) = \text{Sin}\,\theta$

or $2A = \pi - 2B$

or $2(A+B) = \pi$

$A+B = \frac{\pi}{2}$ which shows that the $\triangle$ is rt. $\angle$d $\triangle$.

8.2.1. *Napier's Analogies*

To prove that in any triangle ABC,

(*i*) $\tan\dfrac{B-C}{2}=\dfrac{b-c}{b+c}\operatorname{Cot}\dfrac{A}{2}$

(*ii*) $\tan\dfrac{C-A}{2}=\dfrac{c-a}{c+a}\operatorname{Cot}\dfrac{B}{2}$

(*iii*) $\tan\dfrac{A-B}{2}=\dfrac{a-b}{a+b}\operatorname{Cot}\dfrac{C}{2}$

Proof :—We shall prove only one of these analogies : (*i*) (Say), The remaining two can be proved similarly.

(*i*) $\tan\dfrac{B-C}{2}=\dfrac{b-c}{b+c}\operatorname{Cot}\dfrac{A}{2}$

This is the same thing as :

$$\frac{\tan\dfrac{B-C}{2}}{\operatorname{Cot}\dfrac{A}{2}}=\frac{b-c}{b+c}$$

$$\text{or } \tan\frac{B-C}{2}\tan\frac{A}{2}=\frac{b-c}{b+c} \qquad \left(\because \frac{1}{\operatorname{Cot}\dfrac{A}{2}}=\tan\frac{A}{2}\right)$$

$$\text{R. H. S.}=\frac{b-c}{b+c}$$

$$=\frac{K\operatorname{Sin}B-K\operatorname{Sin}C}{K\operatorname{Sin}B+K\operatorname{Sin}C}$$

$$=\frac{\operatorname{Sin}B-\operatorname{Sin}C}{\operatorname{Sin}B+\operatorname{Sin}C}=\frac{2\operatorname{Cos}\dfrac{B+C}{2}\operatorname{Sin}\dfrac{B-C}{2}}{2\operatorname{Sin}\dfrac{B+C}{2}\operatorname{Cos}\dfrac{B-C}{2}}$$

$$=\frac{\operatorname{Cos}\left(\dfrac{\pi}{2}-\dfrac{A}{2}\right)\operatorname{Sin}\dfrac{B-C}{2}}{\operatorname{Sin}\left(\dfrac{\pi}{2}-\dfrac{A}{2}\right)\operatorname{Cos}\dfrac{B-C}{2}}=\frac{\operatorname{Sin}\dfrac{A}{2}\operatorname{Sin}\dfrac{B-C}{2}}{\operatorname{Cos}\dfrac{A}{2}\operatorname{Cos}\dfrac{B-C}{2}}$$

$$=\frac{\operatorname{Sin}\frac{B-C}{2}}{\operatorname{Cos}\frac{B-C}{2}}\cdot\frac{\operatorname{Sin}\frac{A}{2}}{\operatorname{Cos}\frac{A}{2}}=\tan\frac{B-C}{2}\tan\frac{A}{2}$$

$$=\text{L. H. S.}$$

8.3. *Projection Formulae :*

In any triangle ABC, show that :—

(*i*) $a=b\operatorname{Cos}C+c\operatorname{Cos}B$

(*ii*) $b=c\operatorname{Cos}A+a\operatorname{Cos}C$

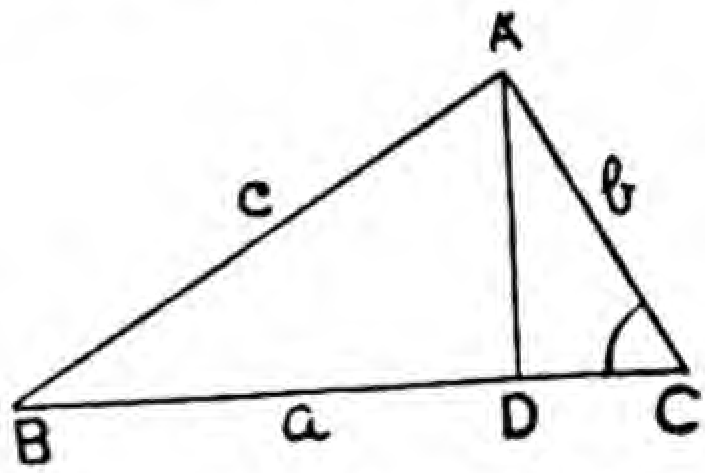
and (*iii*) $c=a\operatorname{Cos}B+b\operatorname{Cos}A$

Proof : (*i*) $a=b\operatorname{Cos}C+c\operatorname{Cos}B$

R. H. S. $=b\operatorname{Cos}C+c\operatorname{Cos}B$
$=K(\operatorname{Sin}B\operatorname{Cos}C+\operatorname{Sin}C\operatorname{Cos}B)$
$=K\operatorname{Sin}(B+C)$
$=K\operatorname{Sin}(180^\circ-A)$
$=K\operatorname{Sin}A=a$

We can slmilarly prove formulae (*ii*) and (*iii*) as well.

Alternative Method (Geometrical)



In the $\triangle$ ABC, draw AD $\perp$ BC

Now $a=BC=BD+DC$

But $\frac{BD}{c}=\operatorname{Sin}B$ $\therefore BD=c\operatorname{Sin}B$

and $\frac{DC}{b}=\operatorname{Sin}C$ $\therefore DC=b\operatorname{Sin}C$

Hence $a=c\operatorname{Sin}B+b\operatorname{Sin}C$

Note : We have taken the triangle ABC as an acute-angled triangle, but the formulae can be derived from any triangle, acute, obtuse, or right-angled.

EXERCISE XII

Prove the following identities in a triangle ABC :—

1. $c \sin\dfrac{A-B}{2} = (a-b)\cos\dfrac{C}{2}$

2. $c \cos\dfrac{A-B}{2} = (a+b)\sin\dfrac{C}{2}$

3. $a\cos A + b\cos B = c\cos(A-B)$

4. $c(\cos A + \cos B\cos C) = b\sin^2 C$

5. $a^2\sin 2B + b^2\sin 2A = 2ab\sin(A+B)$

6. $b^2 - c^2 = a(b\cos C - c\cos B)$

7. $a\cos(B-C) + b\cos(C-A) + c\cos(A-B)$
$= \sin 2A + \sin 2B + \sin 2C$

8. $a\sin(B-C) + b\sin(C-A) + c\sin(A-B) = 0$

9. $\dfrac{a\cos B - b\cos A}{c} = \dfrac{\sin^2 A - \sin^2 B}{\sin^2 C}$

10. $a\sin A + b\sin B - c\sin C = 2a\sin B\sin C$

11. $\dfrac{\sin(A-B)}{\sin(A+B)} = \dfrac{a^2-b^2}{c^2}$

In any triangle ABC, prove that :

12. $\dfrac{a\sin(B-C)}{b^2-c^2} = \dfrac{b\sin(C-A)}{c^2-a^2} = \dfrac{c\sin(A-B)}{a^2-b^2}$

13. $\dfrac{b\sec B + c\sec C}{\tan B + \tan C} = \dfrac{c\sec C + a\sec A}{\tan C + \tan A}$
$= \dfrac{a\sec A + b\sec B}{\tan A + \tan B}$

(*P.U. 1944*)

14. In any $\triangle ABC$, if $\dfrac{\text{Cos A}}{a}=\dfrac{\text{Cos B}}{b}$, Show that the triangle is isosceles.

15. If $C=60^\circ$, show that

$$\frac{1}{a+c}+\frac{1}{b+c}=\frac{3}{a+b+c}$$

16. In a $\triangle$ ABC, show that

$$\frac{b^2-c^2}{a^2}\text{Sin } 2A+\frac{c^2-a^2}{b^2}\text{ Sin } 2B+\frac{a^2-b^2}{c^2}\text{ Sin } 2C=0$$

17. If the sides a, b, c of a $\triangle$ ABC are in A.P., show that

$$\text{Cos }\frac{B-C}{2}=2\text{ Sin }\frac{A}{2}$$

18. If $\angle A=60^\circ$, prove that

$$b+c=2a\text{ Cos }\frac{B-C}{2}$$

8·4. *Cosine Formula*

In any triangle ABC, show that :

(*i*) $\text{Cos A}=\dfrac{b^2+c^2-a^2}{2bc}$

(*ii*) $\text{Cos B}=\dfrac{c^2+a^2-b^2}{2ca}$

and (*iii*) $\text{Cos C}=\dfrac{a^2+b^2-c^2}{2ab}$

Where a, b, c denote the sides of BC, CA, and AB respectively.

Proof : (*i*) We have to prove that

$$\text{Cos A}=\frac{b^2+c^2-a^2}{2bc}$$

We will take three different figures, *i.e.* fig. (*i*) in which $\angle A$ is acute, fig. (*ii*) in which $\angle A$ is obtuse, and fig. (*iii*) in which $\angle A=90^\circ$.

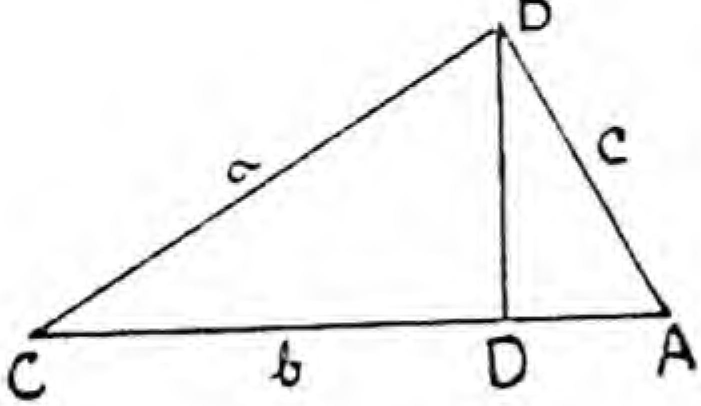


fig. (*i*)

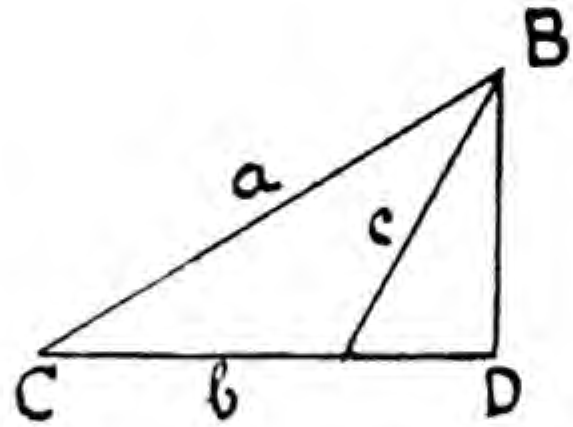


fig. (*ii*)

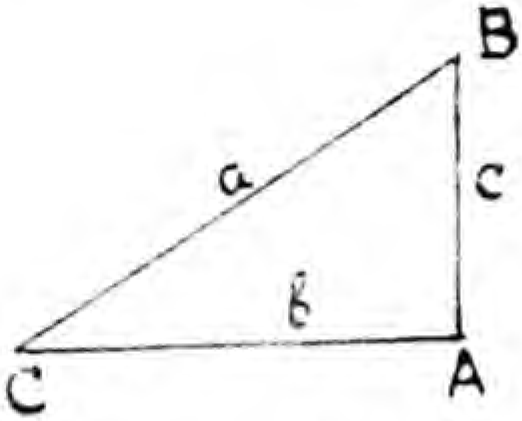


fig. (*iii*)

From the acute-angled $\triangle$ABC (acute-angled at A), we have.

$$BC^2 = CA^2 + AB^2 - 2CA.AD$$

or $a^2 = b^2 + c^2 - 2b.\,AD$

But $\dfrac{AD}{C} = \text{Cos A}$

or $AD = c \text{ Cos A}$

$\therefore\ a^2 = b^2 + c^2 - 2bc \text{ Cos A}$

Hence Cos A

$$= \frac{b^2 + c^2 - a^2}{2bc}$$

From the obtuse-angled $\triangle$ABC (obtuse-angled at A), we have.

$$BC^2 = CA^2 + AB^2 + 2\ CA.AD$$

or $a^2 = b^2 + c^2 + 2b.\ AD$

But $\dfrac{AD}{C}$

$= \text{Cos}(\pi - A)$

$= -\text{Cos A}$

$\therefore\ AD = -C \text{ Cos A}$

$\therefore\ a^2 = b^2 + c^2 - 2bc \text{ Cos A}$

Hence Cos A

$$= \frac{b^2 + c^2 - a^2}{2bc}$$

From the right-angled $\triangle$ ABC (right-angled at A) we have

$$BC^2 = CA^2 + AB^2$$

or $a^2 = b^2 + c^2$

$= b^2 + c^2 - 2bc \text{ Cos } 90°$

($\because$ Cos 90° = 0)

$= b^2 + c^2 - 2bc \text{ Cos A}$

($\because$ Cos A = Cos 90°)

Hence Cos A

$$= \frac{b^2 + c^2 - a^2}{2bc}$$

8.5 Deduction of (*i*) *Cosine* formula and (*ii*) *Projection* for mula from *Sine* formula

Sol. (i) $b^2+c^2-a^2=K^2(\text{Sin}^2 B+\text{Sin}^2 C-\text{Sin}^2 A)$

$=K^2\{\text{Sin}^2 B+\text{Sin}(C+A)\,\text{Sin}(C-A)\}$

$=K^2\{\text{Sin}^2 B+\text{Sin } B\,\text{Sin}(C-A)\}$

$(\because C+A=\pi-B)$

$=K^2\,\text{Sin } B\,\{\text{Sin } B+\text{Sin}(C-A)\}$

$=K^2\,\text{Sin } B\,\{\text{Sin}(C+A)+\text{Sin}(C-A)\}$

$[\because B=\pi-(C+A)]$

$=K^2\,\text{Sin } B\,\{2\,\text{Sin } C\,\text{Cos } A\}$

$=K\,\text{Sin } B\,.\,2\,K\,\text{Sin } C\,.\,\text{Cos } A$

(Please note this step)

$=b\,.\,2\,.\,c\,\text{Cos } A$ $(\because b=k\,\text{Sin } B$ and $c=k\,\text{Sin } C)$

$=2\,bc\,\text{Cos } A$

$\therefore \quad \text{Cos } A=\dfrac{b^2+c^2-a^2}{2bc}$ Similarly, Cos B and Cos C can be found.

(*ii*) This has already been deduced in article 8·3.

8.5.1. Deduction of (*i*) *Sine* formula, and *projection* formula from *Cosine* formula.

Sol. (*i*) We have $\dfrac{a}{\text{Sin } A}=\dfrac{a}{\sqrt{1-\text{Cos}^2 A}}$

$$=\frac{a}{\sqrt{1-\left(\dfrac{b^2+c^2-a^2}{2bc}\right)^2}}$$ (By Cosine formula)

$$=\frac{2abc}{\sqrt{(2bc+b^2+c^2-a^2)(2bc-b^2-c^2+a^2)}}$$

$$=\frac{2abc}{\sqrt{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}}$$

The R. H. S. is a symmetrical expression in a, b, c. We can similarly show that $\dfrac{b}{\text{Sin } B}$ and $\dfrac{c}{\text{Sin } C}$ are each equal to the same

symmetrical expression. Hence $\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}$

(*ii*) We have

$$b\cos C+c\cos B=b\,\frac{a^2+b^2-c^2}{2ab}+c\cdot\frac{a^2+c^2-b^2}{2ca}=\frac{2a^2}{2a}=a$$

$\therefore\ b\cos C+c\cos B=a$

Similarly, we can show that

$$c\cos A+a\cos C=b$$

and $$a\cos B+b\cos A=c$$

EXERCISE XIII

In any triangle ABC, prove that :—

1. $\dfrac{\cos A}{a}+\dfrac{\cos B}{b}+\dfrac{\cos C}{c}=\dfrac{a^2+b^2+c^2}{2abc}$

2. $(a^2-b^2+c^2)\tan B=(a^2+b^2-c^2)\tan C$

3. $\dfrac{b^2-c^2}{a}\cos A+\dfrac{c^2-a^2}{b}\cos B+\dfrac{a^2-b^2}{c}\cos C=0$

4. $2(bc\cos A+ca\cos B+ab\cos C)=a^2+b^2+c^2$

   (**Hint** :—This is the same as Q.1)

5. $\dfrac{b^2-c^2}{a^2}\sin 2A+\dfrac{c^2-a^2}{b^2}\sin 2B+\dfrac{a^2-b^2}{c^2}\sin 2C=0$

6. $4\left(bc\cos^2\dfrac{A}{2}+ca\cos^2\dfrac{B}{2}+ab\cos^2\dfrac{C}{2}\right)=(a+b+c)^2$

   $\left[\textbf{Hint}:\ 2\cos^2\dfrac{A}{2}=1+\cos A \text{ etc.,}\right]$

7. $2\left[a\sin^2\dfrac{C}{2}+c\sin^2\dfrac{A}{2}\right]=c+a-b$

   $\left[\textbf{Hint}:—2\sin^2\dfrac{C}{2}=1-\cos C \text{ etc.}\right]$

8. $(b+c)\cos A+(c+a)\cos B+(a+b)\cos C=a+b+c$

9. $a(b\cos C-c\cos B)=b^2-c^2$

10. In the triangle ABC, BC=14″ CA=15″ and AB=13″. Without using tables, find the values of Cos C, Sin C and Sin A. *(K. U. Pre. 1962)*

[**Ans** : Cos C$=\frac{3}{5}$, Sin C$=\frac{4}{5}$ Sin A$=\frac{56}{65}$]

8.6. *Trigonometrical Ratios of half the angles in terms of sides*

(*a*) *Sines of half the angles.*

We know that $\text{Cos A}=\dfrac{b^2+c^2-a^2}{2bc}$

But $\text{Cos A}=1-2\,\text{Sin}^2\dfrac{A}{2}$

$$\therefore\ 2\,\text{Sin}^2\frac{A}{2}=1-\text{Cos A}=1-\frac{b^2+c^2-a^2}{2bc}$$

$$=\frac{2bc-b^2-c^2+a^2}{2bc}$$

$$\text{or Sin}^2\frac{A}{2}=\frac{a^2-(b^2-2bc+c^2)}{4bc}$$

$$=\frac{a^2-(b-c)^2}{4bc}=\frac{(a+b-c)(a-b+c)}{4bc} \qquad (i)$$

Put $a+b+c=2s$

$\therefore\ a+b-c=2(s-c)$

and $a-b+c=2(s-b)$

$$\therefore (i) \text{ gives, Sin}^2\frac{A}{2}=\frac{2(s-c).\,2(s-b)}{4bc}$$

$$=\frac{(s-b)(s-c)}{bc}$$

$$\therefore \qquad \text{Sin}\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}$$

$$\text{Similarly, Sin}\frac{B}{2}=\sqrt{\frac{(s-c)(s-a)}{ca}}$$

$$\text{and} \qquad \text{Sin}\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{ab}}$$

(*b*) *Cosines of half the angles.*

$$\text{Cos A} = 2\,\text{Cos}^2\frac{A}{2} - 1 \qquad \ldots\ldots(i)$$

$$\text{But Cos A} = \frac{b^2+c^2-a^2}{2bc}$$

$\therefore$ from (*i*) we have

$$2\,\text{Cos}^2\frac{A}{2} - 1 = \text{Cos A} = \frac{b^2+c^2-a^2}{2bc}$$

$$\text{or } 2\,\text{Cos}^2\frac{A}{2} = \frac{b^2+c^2-a^2}{2bc} + 1$$

$$= \frac{b^2+c^2-a^2+2bc}{2bc}$$

$$= \frac{(b+c)^2-a^2}{2bc}$$

$$\text{or } \text{Cos}^2\frac{A}{2} = \frac{(b+c)^2-a^2}{4bc}$$

$$= \frac{(b+c+a)(b+c-a)}{4bc} \qquad \ldots\ldots(ii)$$

Now put $a+b+c=2s$

$\therefore\ b+c-a=2(s-a)$

$\therefore$ From (*ii*) we have

$$\text{Cos}^2\frac{A}{2} = \frac{2s.\,2(s-a)}{4bc}$$

$$= \frac{s(s-a)}{bc}$$

$$\therefore\ \text{Cos}\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$$

$$\text{Similarly, Cos}\frac{B}{2} = \sqrt{\frac{s\,(s-b)}{ca}}$$

$$\text{and Cos}\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$$

(c) *Tangents of half the angles.*

$$\tan\frac{A}{2}=\frac{\operatorname{Sin}\frac{A}{2}}{\operatorname{Cos}\frac{A}{2}}=\frac{\sqrt{\frac{(s-b)(s-c)}{bc}}}{\sqrt{\frac{s(s-a)}{bc}}}$$

[Results obtained in (a) and (b)]

$$=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

Similarly, $\tan\frac{B}{2}=\sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$

and $\tan\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$

Another Method : (Direct Method)

$$\operatorname{Cos} A=\frac{b^2+c^2-a^2}{2bc}\quad \text{But } \operatorname{Cos}A=\frac{1-\tan^2\frac{A}{2}}{1+\tan^2\frac{A}{2}}$$

$$\therefore\quad \frac{1-\tan^2\frac{A}{2}}{1+\tan^2\frac{A}{2}}=\frac{b^2+c^2-a^2}{2bc}$$

$$\text{or}\quad \tan^2\frac{A}{2}=\frac{2bc-b^2-c^2+a^2}{2bc+b^2+c^2-a^2}=\frac{a^2-(b-c)^2}{(b+c)^2-a^2}$$

$$=\frac{(a+b-c)(a-b+c)}{(a+b+c)(b+c-a)}=\frac{2(s-c).2(s-b)}{2s.2(s-a)}$$

(Putting $a+b+c=2s$)

$$=\frac{(s-b)(s-c)}{s(s-a)}$$

$$\therefore\quad \tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

Note :—We have taken the radicals with the *positive* sign, because $\frac{A}{2}$, $\frac{B}{2}$, $\frac{C}{2}$ are all *acute.*

8.7. *To find the Sine of any angle in terms of the sides of a triangle.*

Here we have

$$\text{Sin } A = 2 \text{ Sin } \frac{A}{2} \text{ Cos } \frac{A}{2}$$

$$= 2\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\text{[illegible]}}$$

$$= \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$$

Similarly, $\text{Sin } B = \frac{2}{ca}\sqrt{s(s-a)(s-b)(s-c)}$

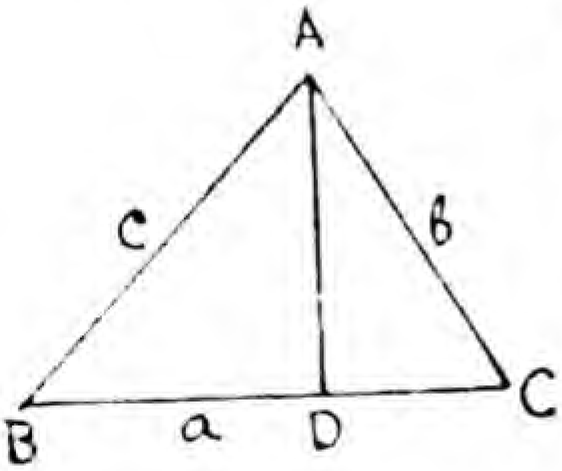
and $\text{Sin } C = \frac{2}{ab}\sqrt{s(s-a)(s-b)(s-c)}$

Cor. From this article it follows that

$$\frac{\text{Sin } A}{a} = \frac{\text{Sin } B}{b} = \frac{\text{Sin } C}{c}$$

$$= \frac{2\sqrt{s(s-a)(s-b)(s-c)}}{abc}$$

8.8. *To find the area of a triangle in terms of its sides.*



Let ABC be the given triangle. Draw AD $\perp$ BC. Let us denote the sides BC, CA, AB by a, b, c respectively.

Now area of the triangle

$= \frac{1}{2}$. BC.AD

$= \frac{1}{2}$. a AD

$= \frac{1}{2}$ ab Sin C

$$\left(\because \frac{AD}{b} = \text{Sin } C\right)$$

$$=\tfrac{1}{2}.\ ab.\ 2\ \text{Sin}\frac{C}{2}\ \text{Cos}\ \frac{C}{2}$$

$$=\tfrac{1}{2}.\ ab.\ 2\sqrt{\frac{(s-a)(s-b)}{ab}}\sqrt{\frac{s(s-c)}{ab}}$$

$$=\frac{ab}{ab}\sqrt{s(s-a)(s-b)(s-c)}$$

$$=\sqrt{s(s-a)(s-b)(s-c)}$$

Where $a+b+c=2s$

This formula is known as **Hero's Formula.**

Solved Examples

Ex. 1. In any $\triangle$ABC, prove that :—

$$c\ \text{Cos}^2\frac{B}{2}+b\ \text{Cos}^2\frac{C}{2}=S$$

Sol. L. H. S. $=c\ \text{Cos}^2\ \frac{B}{2}+b\ \text{Cos}^2\frac{C}{2}$

$$=c.\frac{s(s-b)}{ca}+b.\frac{s(s-c)}{ab}$$

$$=\frac{s(s-b)}{a}+\frac{s(s-c)}{a}=\frac{2s^2-sb-sc}{a}$$

$$=\frac{2s^2-sb-sc-sa+sa}{a}$$

(Please note this step)

$$=\frac{2s^2-s(a+b+c)+sa}{a}$$

$$=\frac{2s^2-2s^2+sa}{a} \qquad (\because a+b+c=2s)$$

$$=s=\text{R. H. S.}$$

Ex. 2. In any triangle ABC, prove that

$$(b+c-a)\ \text{Sin}\frac{A}{2}=2a\ \text{Sin}\frac{B}{2}\ \text{Sin}\ \frac{C}{2}$$

Sol. R. H. S. $=2a \operatorname{Sin}\frac{B}{2} \operatorname{Sin}\frac{C}{2}$

$$=2a\sqrt{\frac{(s-c)(s-a)}{ca}}\sqrt{\frac{(s-a)(s-b)}{ab}}$$

$$=2(s-a)\sqrt{\frac{(s-b)(s-c)}{bc}}=2(s-a)\operatorname{Sin}\frac{A}{2}$$

$$=(2s-2a)\operatorname{Sin}\frac{A}{2}=(a+b+c-2a)\quad\operatorname{Sin}\frac{A}{2}$$

$$=(b+c-a)\operatorname{Sin}\frac{A}{2}=\text{L. H. S.}$$

Ex. 3. If the sides of a triangle are in A. P. prove that $\operatorname{Cot}\frac{A}{2}\operatorname{Cot}\frac{C}{2}=3$

Sol. We have $\operatorname{Cot}\frac{A}{2}\cdot\operatorname{Cot}\frac{C}{2}=3$

$$\therefore \sqrt{\frac{s(s-a)}{(s-b)(s-c)}}\sqrt{\frac{s(s-c)}{(s-a)(s-b)}}=3$$

or $\frac{s}{s-b}=3$

or $3s-3b=s$

or $2s=3b$

or $a+b+c=3b$ $\qquad (\because\ 2s=a+b+c)$

or $\quad a+c=2b$ which is true.

Ex. 4. If a^2, b^2, c^2 are in A. P., show that Cot A, Cot B, Cot C are also in A. P.

Sol. Cot A, Cot B, Cot C will be in A. P.

if Cot A+Cot C=2 Cot B

if $\frac{\operatorname{Cos} A}{\operatorname{Sin} A}+\frac{\operatorname{Cos} C}{\operatorname{Sin} C}=2\frac{\operatorname{Cos} B}{\operatorname{Sin} B}$

if $\dfrac{b^2+c^2-a^2}{2bc.\,ak}+\dfrac{a^2+b^2-c^2}{2ab.\,ck}=2\,\dfrac{c^2+a^2-b^2}{2ca.\,bk}$

$$\left[\because \frac{\text{Sin A}}{a}=\frac{\text{Sin B}}{b}=\frac{\text{Sin C}}{c}=k \text{ (say)}\right]$$

if $(b^2+c^2-a^2)+(a^2+b^2-c^2)=2(c^2+a^2-b^2)$

if $2b^2=2c^2+2a^2-2b^2$

if $4b^2=2c^2+2a^2$

if $2b^2=c^2+a^2$

which is true.

Ex. 5. If Cos A $=$ Sin B $-$ Cos C, prove that the $\triangle$ ABC is a rt. $\angle$d $\triangle$.

Sol. We have Cos A $+$ Cos C $=$ Sin B

or $2 \cos \dfrac{A+C}{2} \cos \dfrac{A-C}{2}=2 \sin \dfrac{B}{2} \cos \dfrac{B}{2}$

or $2 \cos \left(90^\circ-\dfrac{B}{2}\right) \cos \dfrac{A-C}{2}=2 \sin \dfrac{B}{2} \cos \dfrac{B}{2}$

or $\quad 2 \sin \dfrac{B}{2} \cos \dfrac{A-C}{2}=2 \sin \dfrac{B}{2} \cos \dfrac{B}{2}$

or $\cos \dfrac{A-C}{2}=\cos \dfrac{B}{2}$

or $\quad \dfrac{A-C}{2}=\dfrac{B}{2}$

or $\quad A=B+C$

Hence $\angle A=90^\circ$, which proves the question.

EXERCISE XIV

In a $\triangle$ ABC, prove that

1. (*i*) $s=a \cos^2 \dfrac{B}{2}+b \cos^2 \dfrac{A}{2}$

$\quad =b \cos^2 \dfrac{C}{2}+c \cos^2 \dfrac{B}{2}$

$$= c\ \text{Cos}^2 \frac{A}{2} + a\ \text{Cos}^2 \frac{C}{2}$$

2. $s - c = a\ \text{Sin}^2 \frac{B}{2} + b\ \text{Sin}^2 \frac{A}{2}$

3. $bc\ \text{Cos}^2 \frac{A}{2} + ca\ \text{Cos}^2 \frac{B}{2} + ab\ \text{Cos}^2 \frac{C}{2} = s^2$

4. $\dfrac{\text{Cos}^2 \frac{A}{2}}{a} + \dfrac{\text{Cos}^2 \frac{B}{2}}{b} + \dfrac{\text{Cos}^2 \frac{C}{2}}{c} = \dfrac{s^2}{abc}$

5. $\dfrac{\text{Cot} \frac{B}{2}}{\text{Cot} \frac{C}{2}} = \dfrac{a-b+c}{a+b-c}$

6. $\dfrac{\tan \frac{A}{2} - \tan \frac{B}{2}}{\tan \frac{A}{2} + \tan \frac{B}{2}} = \dfrac{a-b}{c}$

7. $\dfrac{\text{Cot} \frac{A}{2} - \text{Cot} \frac{B}{2}}{\text{Cot} \frac{B}{2} - \text{Cot} \frac{C}{2}} = \dfrac{a-b}{b-c}$

8. $(s-a) \tan \frac{A}{2} = (s-b) \tan \frac{B}{2} = (s-c) \tan \frac{C}{2}$

9. $s\left(\tan \frac{B}{2} + \tan \frac{C}{2}\right) = a\ \text{Cot} \frac{A}{2}$

$$= (s-a)\left[\text{Cot} \frac{B}{2} + \text{Cot} \frac{C}{2}\right]$$

10. In a $\triangle$ ABC, if $2b = c + a$, prove that

$$2\ \text{Cot} \frac{B}{2} = \text{Cot} \frac{A}{2} + \text{Cot} \frac{C}{2}$$

11. If in a $\triangle$ ABC, $a \operatorname{Cos}^2 \frac{C}{2} + c \operatorname{Cos}^2 \frac{A}{2} = \frac{3b}{2}$ show that its sides are in AP.

12. If a, b, c are in H. P., prove that

$\operatorname{Sin}^2 \frac{A}{2}, \operatorname{Sin}^2 \frac{B}{2}, \operatorname{Sin}^2 \frac{C}{2}$ are also in H. P.

[**Hint** :—Start with $\operatorname{Sin}^2 \frac{B}{2} = \dfrac{2 \operatorname{Sin}^2 \frac{A}{2} \operatorname{Sin}^2 \frac{C}{2}}{\operatorname{Sin}^2 \frac{A}{2} + \operatorname{Sin}^2 \frac{C}{2}}$

13. If $\tan \frac{A}{2}$, $\tan \frac{B}{2}$, $\tan \frac{C}{2}$ be in A. P. show that Cos A, Cos B, Cos C are also in A. P.

14. If $a+b=3c$, prove that :—

(*i*) $\operatorname{Sin} \frac{A}{2} \operatorname{Sin} \frac{B}{2} = \operatorname{Sin} \frac{C}{2}$

(*ii*) $\operatorname{Cot} \frac{A}{2} \operatorname{Cot} \frac{B}{2} = 2$

(*iii*) $\operatorname{Cot} \frac{A}{2} + \operatorname{Cot} \frac{B}{2} = \operatorname{Cot} \frac{C}{2}$

15. In a $\triangle$ ABC, rt. angled at C, prove that :—

(*i*) $\tan \frac{A-B}{2} = \frac{a-b}{a+b}$ (*ii*) $\operatorname{Sin}^2 \frac{B}{2} = \frac{c-a}{2c}$

(*iii*) $\operatorname{Cos}^2 \frac{A}{2} = \frac{b+c}{2c}$ (*iv*) $\tan \frac{A}{2} = \frac{a-b+c}{a+b+c}$

(*v*) $a \left((1+\tan \frac{B}{2}\right) = \left(1-\tan \frac{B}{2}\right)(b+c)$

CHAPTER IX

Properties of Triangles

9. 1. To prove that $\dfrac{a}{\text{Sin } A} = \dfrac{b}{\text{Sin } B} = \dfrac{c}{\text{Sin } C} = 2R$ when R is the radius of the Circumcircle of the triangle ABC.

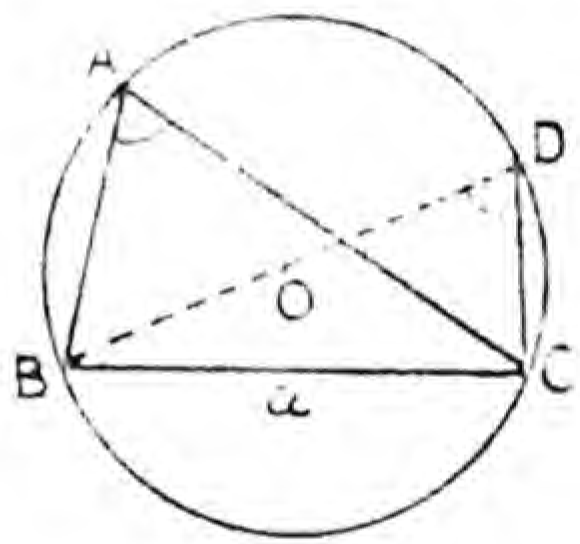


Fig. (i)

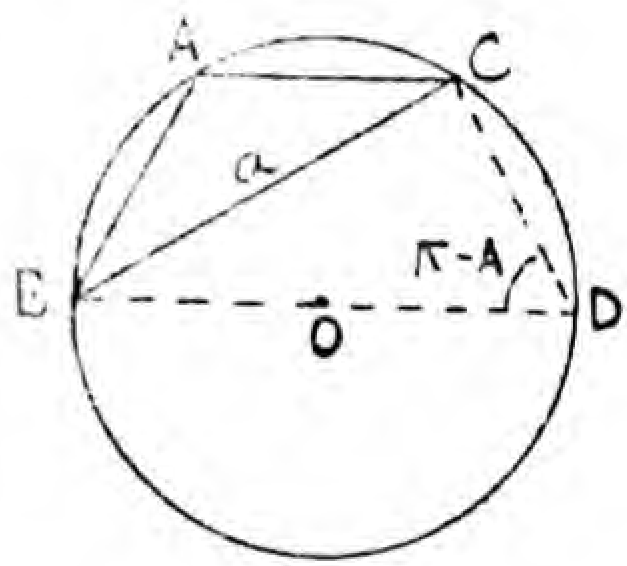


Fig. (ii)

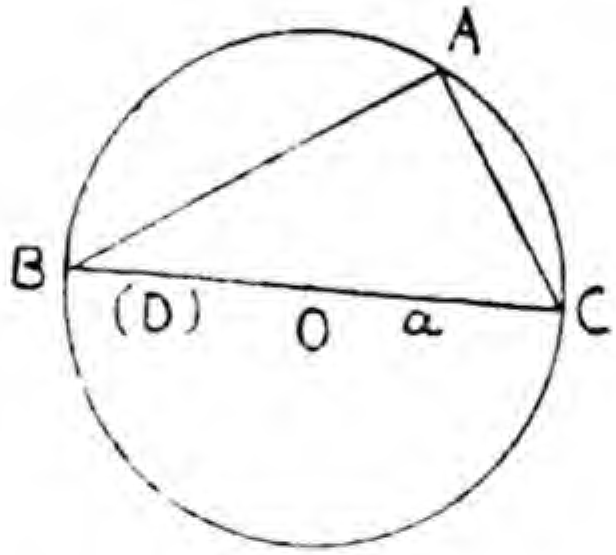


Fig. (iii)

Proof : Let O be the circumcentre. Join BO and produce it to meet the circumcircle at D. Join CD.

In fig. (*i*) $\angle A$ is acute, $\angle A=\angle D$ because these are the angles in the same segment. In fig. (*ii*) $\angle A$ is obtuse, and $\angle D=\pi-A$ because ABDC is a cyclic quadrilateral. In fig. (*iii*) $a=2R$, and $\angle A=90°$

Now, in the $\triangle BCD$ in fig (*i*)

$$\text{Sin } D=\frac{BC}{BD} \qquad (\because \angle BCD=90°)$$

or $$\text{Sin } A=\frac{a}{2R} \qquad \because \angle D=\angle A \text{ and } BD=2R$$

In the $\triangle BCD$ in fig. (*ii*).

$$\text{Sin } D=\frac{BC}{BD}$$

or $$\text{Sin } (\pi-A)=\frac{a}{2R} \qquad \therefore (\angle D=\pi-A)$$

or $$\text{Sin } A=\frac{a}{2R}$$

Lastly, in the $\triangle$ ABC in fig (*iii*)

$$\frac{BC}{DC}=1 \qquad (\because \text{B \& D coincide})$$

or $$\frac{a}{2R}=1=\text{Sin } 90°$$

$$=\text{Sin } A \qquad (\because \angle A=90°)$$

Similarly, it can be shown that

$$\frac{b}{\text{Sin } B}=2R \text{ and } \frac{c}{\text{Sin } C}=2R$$

Hence $$\frac{a}{\text{Sin } A}=\frac{b}{\text{Sin } B}=\frac{c}{\text{Sin } C}=2R$$

Another Expression

To prove that $R = \dfrac{abc}{4\triangle}$ where $\triangle$ denotes the area of the triangle ABC

We know that

$$\frac{a}{\text{Sin A}} = \frac{b}{\text{Sin B}} = \frac{c}{\text{Sin C}} = 2R$$

Take $\dfrac{a}{\text{Sin A}} = 2R$ ......(*i*)

Now $\triangle = \frac{1}{2}\, bc \text{ Sin A}$ (Article 8·8)

which gives $\text{Sin A} = \dfrac{2\triangle}{bc}$

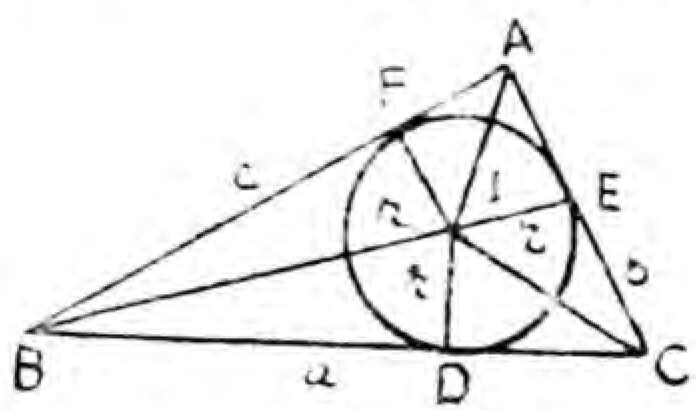
Substituting $\dfrac{2\triangle}{bc}$ for Sin A in (*i*) we get

$$\frac{a}{\dfrac{2\triangle}{bc}} = 2R$$

$$\therefore \qquad R = \frac{abc}{4\triangle}$$

9.2. Radius of the in-Circle

To prove that $r = \dfrac{\triangle}{s}$ where r is the radius of the Circle inscribed in the $\triangle$ ABC and $2s = a + b + c$



Let IA, IB, IC be the bisectors of the angles A, B, C respectively of the $\triangle$ ABC. They will be concurrent, and let them meet in I. From I draw ID, IE, and IF perpendiculars to BC, CA, and AB respectively. Then we know that $ID = IE = IF = r$

Now $\triangle ABC = \triangle IBC + \triangle ICA + \triangle IAB$

or $\triangle = \frac{1}{2} ra + \frac{1}{2} rb + \frac{1}{2} rc$

$= \frac{1}{2} r (a+b+c)$

$= \frac{1}{2} r.\ 2s = rs$

$\therefore \quad r = \dfrac{\triangle}{s}$

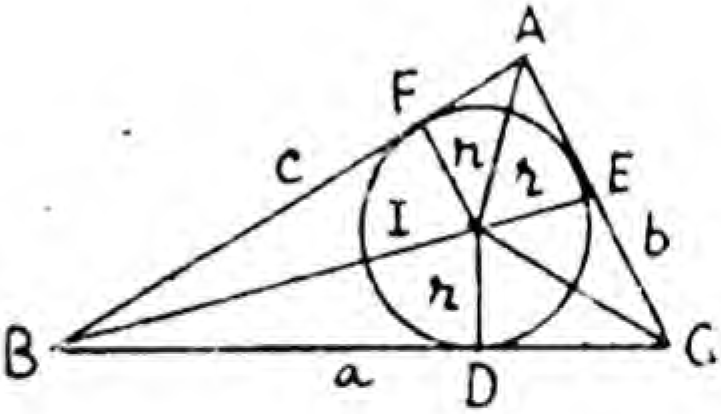
Another Expression

To prove that :—

(i) $r = (s-a) \tan \dfrac{A}{2}$

(ii) $r = (s-b) \tan \dfrac{B}{2}$

(iii) $r = (s-c) \tan \dfrac{C}{2}$



Proof : We know that :—

$2s = a+b+c = BC+CA+AB$

$= (BD+DC)+(CE+EA)+(AF+FB)$

$= (BD+BF)+(DC+CE)+(AE+AF)$

$= 2BD+2DC+2AE$

[$\because$ BD=BF, DC=CE, AE=AF being tangents from external points B, C, D to the circle]

or $2s = 2(BD+DC+AE)$

or $s = (BD+DC)+AE$

$= BC+AE = a+AE$ ......(i)

Now from the rt. angled $\triangle$AIE, we have

$\dfrac{AE}{r} = \text{Cot } \dfrac{A}{2} \qquad \therefore\ AE = r \text{ Cot } \dfrac{A}{2}$

$\therefore$ from (i) we have

$s = a + r \text{ Cot } \dfrac{A}{2}$

Hence $r = (s-a) \tan \dfrac{A}{2}$

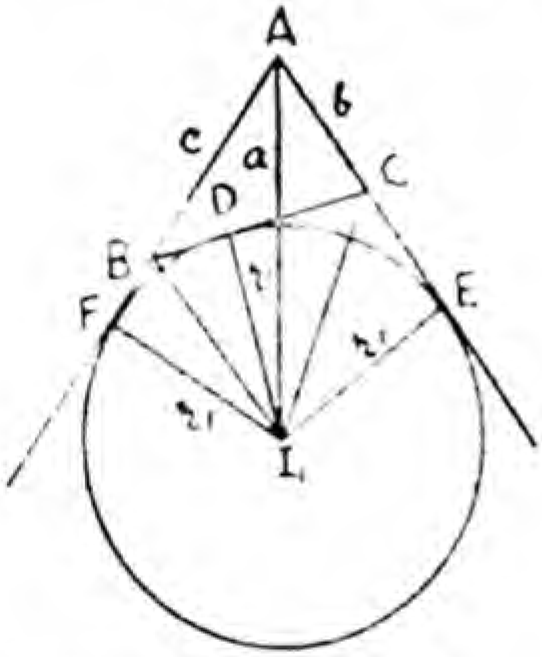
Similarly, we can prove that :—

$$r=(s-b)\tan\frac{B}{2}$$

and $\quad r=(s-c)\quad\tan\dfrac{C}{2}$

9.3. Radii of the escribed circle

To prove that $r_1=\dfrac{\Delta}{s-a}$ where r_1 is the radius of the circle touching the side BC of the $\triangle$ABC.



Let AI_1 be the internal bisector of the angle A and BI_1, CI_1 be the external bisectors of the angles B and C meeting at I_1, the centre of the escribed circle touching the side BC (opposite to A) at D. Draw I_1D, I_1E, and I_1F $\perp$s to BC, and CA and CB (produced) respectively. Let r_1 be the e-radius of the circle touching BC.

Then, $I_1D=I_1E=I_1F=r_1$

Now $\triangle ABC=\triangle I_1AB+\triangle I_1AC-\triangle I_1BC$

or $\quad \Delta=\frac{1}{2}r_1c+\frac{1}{2}r_1b-\frac{1}{2}r_1a$

$\quad\quad =\frac{1}{2}r_1(c+b-a) \quad\quad\quad\quad ......(i)$

Put $a+b+c=2s$

$\therefore\; b+c-a=2(s-a)$

$\therefore$ from (i) we get

$$\Delta=\tfrac{1}{2}.\,r_1.\,2(s-a)$$

Hence $r_1 = \dfrac{\Delta}{s-a}$

Similarly, we can prove that :—

$$r_2 = \frac{\Delta}{s-b} \text{ and } r_3 = \frac{\Delta}{s-c}$$

Where r_2 and r_3 are the radii of the circles touching the sides CA and AB respectively.

Another Expression

To prove that $\mathbf{r_1 = s \tan \frac{A}{2}}$

Proof :— We know that

$$\begin{aligned} 2s &= AB + BC + CA \\ &= AB + (BD + DC) + CA \\ &= (AB + BD) + (DC + CA) \qquad \ldots\ldots(i) \end{aligned}$$

But BD=BF and DC=CE (tangents drawn from external points B and C to the circle with centre I_1)

$\therefore$ From (*i*) we get

$$\begin{aligned} 2s &= (AB + BF) + (AC + CE) \\ &= AF + AE \\ &= 2AF \qquad (\because AE = AF \text{ ; tangents from A}) \qquad \ldots\ldots(ii) \\ \text{or } s &= AF \end{aligned}$$

Now from the rt. angled Δ I_1AF,

$\dfrac{AF}{r_1} = \text{Cot } \dfrac{A}{2}$ $\quad \therefore$ $AF = r_1 \text{ Cot } \dfrac{A}{2}$

$\therefore$ From (*ii*) we get :—

$$S = r_1 \text{ Cot } \frac{A}{2}$$

Which gives $r_1 = s \tan \dfrac{A}{2}$

Similarly, we can Prove that :—

$$r_2 = s \tan \frac{B}{2} \text{ and } r_3 = s \tan \frac{C}{2}$$

Solved Examples.

We give below a number of solved examples. The student is advised to read these carefully. Those marked with an esterisk may be taken as articles.

Ex. 1. Prove that (*i*) $r=\dfrac{a\operatorname{Sin}\frac{B}{2}\operatorname{Sin}\frac{C}{2}}{\operatorname{Cos}\frac{A}{2}}$

and (*ii*) $r=4\text{R}\operatorname{Sin}\frac{A}{2}\operatorname{Sin}\frac{B}{2}\operatorname{Sin}\frac{C}{2}$

Sol. (*i*) R. H. S. $=\dfrac{a\operatorname{Sin}\frac{B}{2}\operatorname{Sin}\frac{C}{2}}{\operatorname{Cos}\frac{A}{2}}$

$$=\frac{a\sqrt{\frac{(s-c)(s-a)}{ca}}\sqrt{\frac{(s-a)(s-b)}{ab}}}{\sqrt{\frac{s(s-a)}{bc}}}$$

$$=a\sqrt{\frac{(s-c)(s-a)}{ca}}\times\sqrt{\frac{(s-a)(s-b)}{ab}}\times\sqrt{\frac{bc}{s(s-a)}}$$

$$=\frac{\sqrt{(s-a)(s-b)(s-c)}}{\sqrt{s}}=\frac{\sqrt{s(s-a)(s-b)(s-c)}}{s}=\frac{\Delta}{s}$$

$$[\because\ \Delta=\sqrt{s(s-a)(s-b)(s-c)}]$$

$=r=$ L. H. S.

(*ii*) R. H. S. $=4\text{R}\operatorname{Sin}\frac{A}{2}\operatorname{Sin}\frac{B}{2}\operatorname{Sin}\frac{C}{2}$

$$=4.\ \frac{abc}{4\Delta}\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{(s-c)(s-a)}{ca}}$$

$$\sqrt{\frac{(s-a)(s-b)}{ab}}$$

$$=\frac{(s-a)(s-b)(s-c)}{\Delta}=\frac{s(s-a)(s-b)(s-c)}{s.\ \Delta}$$

$$=\frac{\Delta^2}{s.\ \Delta}=\frac{\Delta}{s}=r=\text{L. H. S.}$$

Ex. 2. If R and r denote respectively the radii of the circumcircle and incircle of any triangle ABC, prove that

$$\frac{1}{bc}+\frac{1}{ca}+\frac{1}{ab}=\frac{1}{2Rr}$$

(*K.U. 1961*)

Sol. L. H. S. $=\dfrac{1}{bc}+\dfrac{1}{ca}+\dfrac{1}{ab}$

$$=\frac{a+b+c}{abc}=\frac{2s}{abc}$$

$$\text{R. H. S.}=\frac{1}{2Rr}=\frac{1}{2.\frac{abc}{4\Delta}.\frac{\Delta}{s}}=\frac{2s}{abc}$$

$\therefore$ L. H. S. = R. H. S.

Ex. 3. If A, A_1, A_2, A_3 be the areas of the in-circle and three $e-$circles of a triangle, show that

$$\frac{1}{\sqrt{A}}=\frac{1}{\sqrt{A_1}}+\frac{1}{\sqrt{A_2}}+\frac{1}{\sqrt{A_3}}$$

Sol. We have $A=\pi r^2$, $A_1=\pi r_1^2$, $A_2=\pi r_2^2$ and $A_3=\pi r_3^2$

$$\therefore \text{R. H. S.}=\frac{1}{\sqrt{\pi r_1}}+\frac{1}{\sqrt{\pi r_2}}+\frac{1}{\sqrt{\pi r_3}}$$

$$=\frac{1}{\sqrt{\pi}}\left[\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\right]$$

$$=\frac{1}{\sqrt{\pi}}\left[\frac{1}{\frac{\Delta}{s-a}}+\frac{1}{\frac{\Delta}{s-b}}+\frac{1}{\frac{\Delta}{s-c}}\right]$$

$$=\frac{1}{\sqrt{\pi}}\left[\frac{s-a}{\Delta}+\frac{s-b}{\Delta}+\frac{s-c}{\Delta}\right]$$

$$=\frac{1}{\sqrt{\pi}}.\frac{3s-(a+b+c)}{\Delta}=\frac{1}{\sqrt{\pi}}.\frac{s}{\Delta}=\frac{1}{\sqrt{\pi}r}=\frac{1}{\sqrt{A}}$$

Ex 4.* Prove that

$$r_1 = \frac{a \text{ Cos } \frac{B}{2} \text{ Cos } \frac{C}{2}}{\text{Cos } \frac{A}{2}}$$

Sol. R.H.S. $= \dfrac{a \text{ Cos } \frac{B}{2} \text{ Cos } \frac{C}{2}}{\text{Cos } \frac{A}{2}}$

$$= \frac{a \sqrt{\frac{s(s-b)}{ca}} \sqrt{\frac{s(s-c)}{ab}}}{\sqrt{\frac{s(s-a)}{bc}}}$$

$$= a \sqrt{\frac{s(s-b)}{ca}} \sqrt{\frac{s(s-c)}{ab}} \sqrt{\frac{bc}{s(s-a)}}$$

$$= \frac{\sqrt{s(s-b)(s-c)}}{\sqrt{s-a}} = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s-a}$$

(Mark this step)

$$= \frac{\Delta}{s-a} = r_1$$

Ex. 5. Prove that

$$2 R^2 \text{ Sin A Sin B Sin C} = \Delta$$

Sol. L.H.S. $= 2R^2 \,.\, \text{Sin A} \,.\, \text{Sin B} \,.\, \text{Sin C}$

$$= 2 \,.\, R^2 \,.\, \frac{a}{2R} \,.\, \frac{b}{2R} \,.\, \frac{c}{2R}$$

$$\left[\because \frac{a}{\text{Sin A}} = \frac{b}{\text{Sin B}} = \frac{c}{\text{Sin C}} = 2R\right]$$

$$= \frac{abc}{4R} = \frac{abc}{4} \,.\, \frac{4\Delta}{abc} = \Delta = \text{R.H.S.}$$

Ex. 6. Prove that

$$\left(\frac{s}{a}-1\right)\left(\frac{s}{b}-1\right)\left(\frac{s}{c}-1\right)=\frac{r}{4R}$$

Sol. L.H.S. $=\dfrac{S-a}{a}\cdot\dfrac{s-b}{b}\cdot\dfrac{s-b}{c}$

$$=\frac{s(s-a)(s-b)(s-c)}{s\,.\,abc} \quad \text{(Please mark this step)}$$

$$=\frac{\triangle^2}{s\,.\,abc} \qquad [\because \sqrt{s(s-a)(s-b)(s-c)}=\triangle]$$

$$\text{R.H.S.}=\frac{r}{4R}=\frac{\frac{\triangle}{s}}{\frac{4\,.\,abc}{4\triangle}}=\frac{\triangle^2}{s\,.\,abc}=\text{L.H.S.}$$

Ex. 7. Show that $\text{Cos A}+\text{Cos B}+\text{Cos C}=1+\dfrac{r}{R}$

(K.U. 1953)

Sol. $\text{Cos A}+\text{Cos B}+\text{Cos C}=2\,\text{Cos}\,\dfrac{A+B}{2}\,\text{Cos}\,\dfrac{A-B}{2}$

$$+1-2\,\text{Sin}^2\frac{C}{2}$$

$$=1+2\,\text{Cos}\left(90^\circ-\frac{C}{2}\right)\text{Cos}\,\frac{A-B}{2}-2\,\text{Sin}^2\frac{C}{2}$$

$$=1+2\,\text{Sin}\,\frac{C}{2}\,\text{Cos}\,\frac{A-B}{2}-2\,\text{Sin}^2\frac{C}{2}$$

$$=1+2\,\text{Sin}\,\frac{C}{2}\left[\text{Cos}\,\frac{A-B}{2}-\text{Sin}\,\frac{C}{2}\right]$$

$$=1+2\,\text{Sin}\,\frac{C}{2}\left[\text{Cos}\,\frac{A-B}{2}-\text{Sin}\left(90^\circ-\frac{A+B}{2}\right)\right]$$

$$=1+2\,\text{Sin}\,\frac{C}{2}\left[\text{Cos}\,\frac{A-B}{2}-\text{Cos}\,\frac{A+B}{2}\right]$$

$$=1+2\,\text{Sin}\,\frac{C}{2}\left[2\,\text{Sin}\,\frac{A}{2}\,\text{Sin}\,\frac{B}{2}\right]$$

$$=1+4\ \text{Sin}\frac{A}{2}\ \text{Sin}\ \frac{B}{2}\ \text{Sin}\ \frac{C}{2}$$

$$=1+4\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{(s-c)(s-a)}{ca}}$$

$$\sqrt{\frac{(s-a)(s-b)}{ab}}$$

$$=1+4\frac{(s-a)(s-b)(s-c)}{abc}$$

$$=1+4.\ \frac{s(s-a)(s-b)(s-c)}{s\ .\ abc}=1+\frac{4\Delta^2}{s\ .\ abc}$$

$$\text{R.H.S.}=1+\frac{r}{R}=1+\frac{\frac{\Delta}{s}}{\frac{abc}{4\Delta}}=1+\frac{4\Delta^2}{s\ .\ abc}$$

$$=\text{L.H.S}$$

Ex. 8. If p_1, p_2, p_3 are the perpendiculars from the angular points of a triangle to the opposite sides, show that

(*i*) $p_1=\frac{a}{\text{Cot B}+\text{Cot C}}$ (*ii*) $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}=\frac{1}{r}$

and (*iii*) $8\ R^3=\frac{a^2b^2c^3}{p_1p_2p_3}$

Sol. We have $\Delta=\frac{1}{2}\ p_1\ a=\frac{1}{2}\ p_2\ b=\frac{1}{2}\ p_3c.$

$$\therefore \qquad p_1=\frac{2\Delta}{a},\ p_2=\frac{2\Delta}{b},\ p_3=\frac{2\Delta}{c}$$

(*i*) $$p_1=\frac{a}{\text{Cot B}+\text{Cot C}}$$

$$\text{R.H.S.}=\frac{a}{\frac{\text{Cos B}}{\text{Sin B}}+\frac{\text{Cos C}}{\text{Sin C}}}=\frac{a\ \text{Sin B Sin C}}{\text{Cos B Sin C}+\text{Cos C Sin B}}$$

$$=\frac{a\ \text{Sin B Sin C}}{\text{Sin (B+C)}}=\frac{a\ \text{Sin B Sin C}}{\text{Sin}\ (\pi-A)}$$

$$=\frac{a \text{ Sin B Sin C}}{\text{Sin A}}=\frac{a \cdot \frac{b}{2R} \cdot \frac{c}{2R}}{\frac{a}{2R}}$$

$$=\frac{bc}{2R}=\frac{bc}{2 \cdot \frac{abc}{4\triangle}}=\frac{2\triangle}{a}=\text{L.H.S.}$$

(*ii*) $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}=\frac{1}{r}$

$$\text{L.H.S.}=\frac{a}{2\triangle}+\frac{b}{2\triangle}+\frac{c}{2\triangle}$$

$$=\frac{1}{2\triangle}(a+b+c)=\frac{2s}{2\triangle}=\frac{s}{\triangle}=\frac{1}{r}$$

(*iii*) $8\,R^3=\frac{a^2b^2c^2}{p_1p_2p_3}$

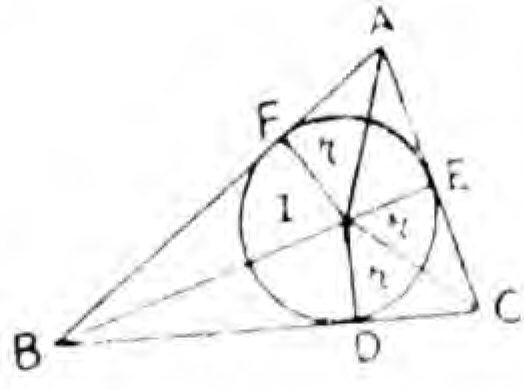
$$\text{R.H.S.}=\frac{a^2b^2c^2}{p_1p_2p_3}=\frac{a^2b^2c^2}{\frac{2\triangle}{a} \cdot \frac{2\triangle}{b} \cdot \frac{2\triangle}{c}}=\frac{a^3b^3c^3}{8\triangle^3}$$

$$=8\left(\frac{abc}{4\triangle}\right)^3=8\,.\,R^3$$

$$\left(\because \frac{abc}{4\triangle}=R\right)$$

Ex. 9. If α, β, γ be the distances of the vertices of a $\triangle$ from the incentre, prove that $\alpha\beta\gamma s=abcr$

Sol. As is clear from the figure,



$\alpha=IA=r \text{ Cosec } \frac{A}{2}$

$\beta=IB=r \text{ Cosec } \frac{B}{2}$

and $\gamma=IC=r \text{ Cosec } \frac{C}{2}$

Now, L.H.S $=\alpha\beta\gamma s$

$$= r \operatorname{Cosec} \frac{A}{2} \cdot r \operatorname{Cosec} \frac{B}{2} \cdot r \operatorname{Cosec} \frac{C}{2} \cdot s$$

$$= r^3 \sqrt{\frac{bc}{(s-b)(s-c)}} \sqrt{\frac{ac}{(s-c)(s-a)}} \sqrt{\frac{ab}{(s-a)(s-b)}} \cdot s$$

$$= r^3 \cdot \frac{abc \cdot s}{(s-a)(s-b)(s-c)}$$

$$= \frac{\triangle^3}{s^3} \cdot \frac{abc \cdot s^2}{s(s-a)(s-b)(s-c)} = \frac{\triangle^3}{s} \cdot \frac{abc}{\triangle^2}$$

$$= \frac{\triangle}{s} \cdot abc = abc \cdot r = \text{R.H.S.}$$

Ex. 10. Prove that

$$R = \frac{abc\,(\operatorname{Cot} A + \operatorname{Cot} B + \operatorname{Cot} C)}{a^2+b^2+c^2}$$

Sol. R.H.S. $= \dfrac{abc\,(\operatorname{Cot} A + \operatorname{Cot} B + \operatorname{Cot} C)}{a^2+b^2+c^2}$

$$= \frac{abc}{a^2+b^2+c^2}\left[\frac{\operatorname{Cos} A}{\operatorname{Sin} A} + \frac{\operatorname{Cos} B}{\operatorname{Sin} B} + \frac{\operatorname{Cos} C}{\operatorname{Sin} C}\right]$$

$$= \frac{abc}{a^2+b^2+c^2}\left\{\frac{\frac{b^2+c^2-a^2}{2bc}}{\frac{a}{2R}} + \frac{\frac{c^2+a^2-b^2}{2ca}}{\frac{b}{2R}} + \frac{\frac{a^2+b^2-c^2}{2ab}}{\frac{c}{2R}}\right\}$$

$$= \frac{abc}{a^2+b^2+c^2} \cdot \left[\frac{2R\,(b^2+c^2-a^2)}{2\,abc} + \frac{2R(c^2+a^2-b^2)}{2abc} + \frac{2R\,(a^2+b^2-c^2)}{2abc}\right]$$

$$= \frac{abc}{a^2+b^2+c^2} \cdot \frac{2R}{2abc}\,(a^2+b^2+c^2) = R = \text{L.H.S}$$

EXERCISE XV

In any $\triangle$ ABC, prove that :—

1. $4Rrs = abc$. 2. $Rr(\text{Sin } A+\text{Sin } B+\text{Sin } C)=\triangle$.

3. $rr_1 = r_2\, r_3 \tan^2 \frac{A}{2}$. 4. $\tan\frac{A}{2} = \frac{rr_1}{\triangle}$

5. $r^3 \text{Cot}^2 \frac{A}{2} \text{Cot}^2 \frac{B}{2} \text{Cot}^2 \frac{C}{2} = r_1\, r_2\, r_3$.

6. $\frac{1}{r^2} + \frac{1}{{r_1}^2} + \frac{1}{{r_2}^2} + \frac{1}{{r_3}^2} = \frac{a^2+b^2+c^2}{\triangle^2}$

7. (i) $r\left(\text{Cot } \frac{A}{2} + \text{Cot } \frac{B}{2} + \text{Cot } \frac{C}{2}\right) = s$

(ii) $r = a\text{Sin } \frac{B}{2} \text{Sin } \frac{C}{2} \text{Sec } \frac{A}{2}$

8. $a \text{ Cot } A + b \text{ Cot } B + c \text{ Cot } C = 2R + 2r$.

9. $\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} - \frac{1}{s} = \frac{4r}{\triangle}$

10. $\text{Cos } A + \text{Cos } B + \text{Cos } C = 1 + \frac{r}{R}$ → Example (No. 7)

11. $\text{Sin } A + \text{Sin } B + \text{Sin } C = \frac{s}{R}$

12. $\triangle = 2R^2 \text{ Sin } A \text{ Sin } B \text{ Sin } C$.

13. $a \text{ Cos } A + b \text{ Cos } B + c \text{ Cos } C = 4R \text{ Sin } A \text{ Sin } B \text{ Sin } C$.

14. $\frac{a \text{ Cos } A + b \text{ Cos } B + c \text{ Cos } C}{a+b+c} = \frac{r}{R}$

15. $\text{Cos}^2 \frac{A}{2} + \text{Cos}^2 \frac{B}{2} + \text{Cos}^2 \frac{C}{2} = 2 + \frac{r}{2r}$

16. (i) $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$.

(ii) $r_1 + r_2 + r_3 - r = 4R$

(iii) $r_2 r_3 + r_3 r_1 + r_1 r_2 = s^2$

(iv) $(r_1 - r)(r_2 + r_3) = a^2$

17. (i) $(r_1+r_2)\tan\frac{C}{2}=(r_3-r)\operatorname{Cot}\frac{C}{2}=c$

(ii) $(r_2+r_3)\sqrt{\frac{rr_1}{r_2r_3}}=a$

(iii) $4\triangle\operatorname{Cot}A=b^2+c^2-a^2$

(iv) $rs=r_1(s-a)=r_2(s-b)=r_3(s-c)=\triangle$

(v) $16R^2rr_1r_2r_3=a^2b^2c^2$

18. $\frac{b-c}{r_1}+\frac{c-a}{r_2}+\frac{a-b}{r_3}=0$

19. $\left(\frac{1}{r}-\frac{1}{r_1}\right)\left(\frac{1}{r}-\frac{1}{r_2}\right)\left(\frac{1}{r}-\frac{1}{r_3}\right)=\frac{16}{r^2(a+b+c)^2}$

20. $a=\frac{r_1(r_2+r_3)}{\sqrt{(r_1r_2+r_2r_3+r_3r_1)}}$

21. (i) $\frac{r_1}{bc}+\frac{r_2}{ca}+\frac{r_3}{ab}=\frac{1}{r}-\frac{1}{2R}$

(ii) $(r_1-r)(r_2-r)(r_3-r)=4Rr^2$

22. $\frac{r_2+r_3}{(s-a)\operatorname{Sin}A}=\frac{r_3+r_1}{(s-b)\operatorname{Sin}B}=\frac{r_1+r_2}{(s-c)\operatorname{Sin}C}$

CHAPTER X

Logarithms and Their Use

Definition :

Let a, x, and N be three numbers or quantities related by the equation $a^x = N$, then x is called the logarithm of the number N to the base a, and is denoted as $x = \log_a N$. It is, therefore, the index of the power to which the base must be raised, that it may be equal to the given number. *Log* (abbreviation of logarithm) is an *Operator* which when operated on any number 'N' means, "Find out the number 'x' to which 'a' has to be raised to give the number N." *e.g.* $2^3 = 8$; here 3 the index of the power of 2 is the logarithm of 8 to the base 2, which quantity when developes a power 3 becomes 8. This idea is symbolically expressed as $3 = \log_2 8$.

Ex. 1. (*i*) $a^1 = a$ or $1 = \log_a a$; *i.e.*, 1 is the logarithm of a to the base a. [**Note** : Logarithm of a number to the same number as base is always one].

(*ii*) $a^0 = 1$ or $\log_a 1 = 0$; *i.e.* [Logarithm of 1 is always zero, whatever base be taken.]

(*iii*) $2^4 = 16$ or $\log_2 16 = 4$; also $4^2 = 16$ or $\log_4 16 = 2$, [**Note** :—Logarithms of the same number to different bases are different].

(*iv*) $3^{\cdot 5} = 1{\cdot}73205$ or $\cdot 5 = \log_3 1{\cdot}73205$; $5^{-2} = \frac{1}{25} = \cdot 04$, or $\log_5 \cdot 04 = -2$. [**Note :** Logarithms of numbers greater than one are positive and of numbers less than one are nagative.]

(*v*) $10^3 = 1000$ or $\log_{10} 1000 = 3$; $10^5 = 100000$ or $\log_{10} 100000 = 5$ etc.

[Logarithms of numbers to the same base increase as the numbers increase.]

Ex. 2. Evaluate : (*i*) $\log_9 27$, (*ii*) $\log_4 \cdot 5$, (*iii*) $\log_3 .3$, (*iv*) $\log_a 0$.

[Ans : (*i*) 1·5. (*ii*) −·5 (*iii*) −1, (*iv*) −∞.]

10·2 Theorems on Logarithms :

(*a*) **The Logarithm of a Product of two or more numbers is equal to the sum of the logarithms of its factors.**

First law of indices gives $a^m \times a^n = a^{m+n}$. (*i*)

Let $a^m = x$ & $a^n = y$ *i.e.* $\log_a x = m$ and $\log_a y = n$, (*ii*) [By definition]

Now, (*i*) becomes from (*ii*), $a^{m+n} = xy$ or $\log_a xy = m+n$:

From (*ii*), $\mathbf{\log_a xy = \log_a x + \log_a y}$. (1)

Similarly, $\log_a xyz = \log_a xy + \log_a z$
$= \log_a x + \log_a y + \log_a z$.

Ex. 3. $\log_a 2310 = \log_a (2 \times 3 \times 5 \times 7 \times 11)$
$= \log_a 2 + \log_a 3 + \log_a 5 + \log_a 7 + \log_a 11$.

(*b*) **The logarithm of a quotient of two quantities is equal to the difference of the logarithms of the numerator and the denominator.**

Second Law of Indices gives $\dfrac{a^m}{a^n} = a^{m-n}$. (*i*)

Put as before $a^m = x$ and $a^n = y$.

i.e. $m = \log_a x$, and $n = \log_a y$; (*ii*)

Now, (*i*) with the help of (*ii*), becomes $a^{m-n} = \dfrac{x}{y}$.

From definition, $\log_a \dfrac{x}{y} = m - n$

$= \log_a x - \log_a y$ from (*ii*)

$$\therefore \mathbf{\log_a \frac{x}{y} = \log_a x - \log_a y}. \quad (2)$$

Ex. 4 $\log_{10} \dfrac{15}{38} = \log_{10} 15 - \log_{10} 38$,

$= \log_{10}(3 \times 5) - \log_{10}(2 \times 19)$,
$= \log_{10} 3 + \log_{10} 5 - \log_{10} 2 - \log_{10} 19$.

(c) **The logarithm of any number having a power is equal to the logarithm of the same number multiplied by the index of its power.**

Third law of indices gives $(a^m)^n = a^{mn}$ (i)

Put $a^m = x$ i.e $m = \log_a x$. (ii)

$\therefore$ If $x^n = a^{mn}$. by definition, $mn = \log_a x^n$

or $\mathbf{\log_a x^n = n \log_a x}$. (3)

Ex. 5. $\log_{10} 27 = \log_{10} 3^3 = 3 \log_{10} 3$.

(*d*) **Conversion of a logarithm of a number from one base to the other** :

Let x and y be the logarithms of a number N to bases a and b respectively, then, by definition of logarithms, we have

$$x = \log_a N \qquad \text{or} \qquad a^x = N,$$
$$y = \log_b N \qquad \text{or} \qquad b^y = N,$$
$$\text{i.e., } a^x = b^y. \qquad \text{(i)}$$

Raising both the sides of (*i*) to $\frac{1}{x}$ and $\frac{1}{y}$, we get

$$a = b^{\frac{y}{x}} \text{ and } \qquad b = a^{\frac{x}{y}} \text{ respectively.} \qquad \text{(ii)}$$

Applying the definitions again in (*ii*), we get

$$\frac{y}{x} \log_b a, \ \frac{x}{y} = \log_a b. \qquad \text{(iii)}$$

Multiplying (*iii*) together, we get very important result

$$\log_b a \times \log_a b = \frac{y}{x} \times \frac{x}{y} = 1,$$

$$\therefore \mathbf{\log_b a \times \log_a b = 1} \qquad (4)$$

For conversion, substituting from (*i*) and (*iii*) and applying (4)

$$\frac{\log_b N}{\log_a N} = \frac{y}{x} = \log_b a = \frac{1}{\log_a b},$$

$$\therefore \mathbf{\log_b N = \frac{\log_a N}{\log_a b}}. \qquad (5)$$

Also, $\dfrac{\log_a N}{\log_b N}=\dfrac{x}{y}=\log_a b=\dfrac{1}{\log_b a}$:

$$\therefore\ \log_a N=\frac{\log_b N}{\log_b a}. \tag{6}$$

Ex. 6. $\log_4 3=\log_{10}3\times\log_4 10=\log_{10}3\times\dfrac{1}{\log_{10}4}$.

[$\because \log_4 10\times\log_{10}4=1$]

Note. The readers are requested kindly to note the following mistakes generally committed in taking logarithms ;

(*i*) $\log_a(x+y)=\log_a x+\log_a y$ which is absurd.

(*ii*) $\log_a(x^y+y^x)=y\log_a x+x\log_a y$ which is also fundamentally wrong.

(*iii*) $\log_a 5x=5\log_a x$ in place of $\log_a 5+\log_a x$

EXERCISE XVI

1. Simplify the following and express the results in the logarithmic form :

(*i*) $2^{\cdot 5}$, (*ii*) $27^{-\cdot 5}$, (*iii*) $16^{\cdot 75}$, (*iv*) $256^{-\cdot 25}$.

2. From the definition, show that $x^y=e^{y\log x}$.

[*Alld* *1945*]

3. Evaluate : (*i*) $\log_7 343$, (*ii*) $\log_2 \cdot 5$, (*iii*) $\log_{\cdot 01}100$.
(*iv*) $\log_4 64$.

4. If x, y, z are positive, prove that

$$\log\frac{x^2}{yz}+\log\frac{y^2}{zx}+\log\frac{z^2}{xy}=0.$$

5. Simplify : (*i*) $\log_a\dfrac{\sqrt{4}\times 27^{\frac{1}{3}}}{36^{\frac{1}{2}}}$, (*ii*) $\log_a\left(\dfrac{45^2}{28^3}\div\dfrac{\sqrt{70^4}}{81}\right)$.

(*iii*) $\log_a\dfrac{\sqrt{100\times 65^2}}{628^{-\frac{1}{3}}\times 52^{-3}}$, (*iv*) $\log_a\dfrac{112^{\frac{3}{4}}\div 343^{\frac{1}{5}}}{28^2\times 512^{-\frac{1}{9}}}$

6. If $a=\log\frac{5}{6}$, $b=\log\frac{10}{9}$, $c=\log\frac{25}{24}$, prove that

[*Alld. 1936*]

$\log 2=b+c-3a$.

7. (*a*) Show that $\log_a b\times\log_b c\times\log_c a=1$.

(*b*) $\log_{2a}a=x$, $\log_{3a}2a=y$, $\log_{4a}3a=z$,

show that $xyz+1=2yz$.

10.3 Common logarithms.

It is not always that actual value of logarithm of a number to any base is obtained *exactly*, for instance, the value of $\log_4 181$ lies between 3 and 4 ; since, if $\log_4 181=x$, then $4^x=181$. Considering the multiples of 4 nearest to 181, we observe $4^3<181<4^4$ or $4^3<4^x<4^4$ *i.e.* $3<x<4$. Hence the value is 3 plus some fraction. The *integral part* in the value of the logarithm of a number is called *Characteristic* of the logarithm, and *fractional part* its *Mantissa*. In Logarithmic tables, we get only the fractional part of the value, calculated either to the base '*e*' a transendental number approximately equal to 2.7, or reckoned to the base 10. Logarithms to the former base are called *Natural or Napierian*, while to the latter base *Common Logarithms*. For all practical purposes, it is the *Common system* of logarithms we use, as 10 is mostly adopted as radix in the numerical calculations.

Note :—If no base of a logarithm is mentioned, it should be considered as 10. In practice, common logarithms are expressed without base.

Ex. 7. (*i*) $\log 2340=3\cdot3692$.

[Here 3 is *characteristic* and $\cdot3692$ the decimal positive fraction is *mantissa*.]

(*ii*) $\log \cdot000234=4\cdot3692$.

[Here the characteristic is negative and equal to -4, the mantissa is +ve and equal to $\cdot3692$.

Note : The characteristic may be positive or negative but Negative characteristic is written with a *bar* over it, to separate it from the +ve mantissa.

Ex. 8. Find the characteristic and mantissa when log $\cdot0234 = -1\cdot6308$.

Here $-1\cdot6308 = -1 - \cdot6308 = -2 + 1 - \cdot6308$
$= -2 + \cdot3692 = \bar{2}\cdot3692.$

Hence, -2 is the characteristic and $+\cdot3692$ is the mantissa.

10.4 Advantages of the common system of Logarithms.

(*i*) **The characteristic of the logarithm of any number to the base 10 can be found out by inspection of the number of digits in the integral part, or the number of seros after the decimal point and before the first zignificant digit.**

(*ii*) **The mantissa of logarithms of all numbers consisting of the same digits and in the same order are the same ; i.e. the mantissa remains unchanged if the number is multiplied or divided by any multiple of 10.**

10.5. Case I—The characteristic of all numbers greater than one to the base 10 is positive integer and always one less than the number of digits in the integral part of the numbers.

Let $10^0 = 1$,	*i.e.*	$0 = \log_{10} 1.$	(*i*)
$10^1 = 10.$	*i.e.*	$1 = \log_{10} 10.$	(*ii*)
$10^2 = 100.$	*i.e.*	$2 = \log_{10} 100.$	(*iii*)
$10^3 = 1000.$	*i.e.*	$3 = \log_{10} 1000.$	(*iv*)

(*a*) We observe, from (*i*) and (*ii*), that numbers lying between 1 and 10, *i.e.* having one digit in the integral part, have their logarithms between 0 and 1 (only a +ve droper fraction) Thus, $\log_{10} 3.705 = 0 + f$.

(*b*) From (*ii*) and (*iii*), we find numbers lying between 10 and 100 (*i.e.* two digit-numbers in the integral part) have their logarithms between 1 and 2.

Thus, $\log_{10} 43\cdot201 = 1 + f$.

(c) From (*iii*) and (*iv*), we see a number having three digits in the integral part, lies between 100 and 1000, and has its logarithms between 2 and 3.

Thus, $\log_{10} 758{\cdot}1 = 2 + f$.

(d) And so generalising, a number N ($>$I) having n digits in the integral part lies between 10^{n-1} and 10^n and will have its logarithm between $n-1$ and n.

Thus, $\log_{10} N = (n-1) + f$.

Hence, we have the above rule for numbers greater than one.

10.6. Case II. The characteristic of logarithm to the base 10 of any number less than one is always negative and one more than the number of zeros after the decimal point and before the first significant digit.

Consider the following :

$$10^0 = 1, \text{ i.e. } \log_{10} 1 = 0. \qquad (i)$$

$$10^{-1} = \frac{1}{10} = {\cdot}1, \text{ i.e. } \log_{10} {\cdot}1 = -1. \qquad (ii)$$

$$10^{-2} = \frac{1}{10^2} = {\cdot}0, \text{ i.e. } \log_{10} {\cdot}01 = -2. \qquad (iii)$$

$$10^{-3} = \frac{1}{10^3} = {\cdot}001, \text{ i.e. } \log_{10} {\cdot}001 = -3. \qquad (iv)$$

and so on.

(a) We observe, from (*i*) and (*ii*), that any number lying between 1 and ·1 has *no zero* after the decimal point and before the first significant digit and its logarithm lies between 0 and -1, *i.e* a negative proper fraction which can be expressed as $-1+f$, and its characteristic is -1.

Thus. $\log_{10} {\cdot}2045 = -1 + f$.

(b) From (*ii*) and (*iii*), we see that any number lying between ·1 and ·01 has *one zero* immediately after the decimal point and before the first significant digit and its logarithm lies between -1 and -2 *i.e.* $-2+f$, and its characteristic is -2.

Thus, $\log_{10} .045003 = -2+f$.

(*e*) From (*iii*) and (*iv*), we see that any number lying between. 01 and .001 has *two zeros* immediately after the decimal point and before the first significant digit and its logarithm lies between -2 and -3 *i.e.* $-3+f$, and its characteristic is -3.

Thus. $\log_{10} \cdot 004503 = -3+f$.

(*d*) And so generalising, a number N (<1), having *n* zeros immediately after the *decimal point* and before the *first significant digit*, lies between $10^{-(n+1)}$ and 10^{-n} and will have its logarithm between $-(n+1)$ and $-n$ and characteristic as $-(n+1)$.

Thus, $\log_{10} N = -(n+1)+f$

Note (*i*) When $N=1$, logarithm is *zero*.

Note (*ii*) When N is negative, logarithm is imaginary hence, a negative number has no logarithms.

Note (*iii*) Conventionally, *negative* characteristic and *positive* mantissa is denoted by placing a minus sign above the characteristic.

Thus, $\log N = \overline{\mathbf{n+1}}\,\mathbf{\cdot abcd}$..............where $\mathbf{f} = \mathbf{\cdot abcd}$........,N has **n** zeros between the decimal sign and before the first significant digit.

Ex 9. Find the characteristics of logarithms of

(a) 5·234, (2) ·0043, (c) 421·3, (d) ·2005.

Sol. (*a*) The given number is greater than one, and the number of digits in the integral part is one.

$\therefore$ Characteristic$=1-1=0$.

(*b*) The given number is less than one, and the number of zeros immediately after the decimal point and before the first significant digit is two.

$\therefore$ Characteristic$=-(2+1)=-3$.

(*c*) The given number is greater than one and has three digits in the integral part.

$\therefore$ Characteristic$=3-1=2$.

(*d*) The given number is less than one, and there is no zero after the decimal point and before the first significant digit.

$\therefore$ The characteristic$=-(0+1)=-1$.

EXERCISE XVII

Write, by inspection, the characteristics of the logarithms of the following :

(*i*) 1·523; (*ii*) 305·2; (*iii*) 527000; (*iv*) ·2405 (*v*) ·00201; (*vi*) ·0000070403; (*vii*) ·45000001; (*viii*) ·08; (*ix*) 20 1; (*x*) 200001.

10·7. The mantissa of logarithms of all numbers consisting of the same digits and in the same order is the same.

Proof—All numbers consisting of the same digits and arranged in the same order, only differ in the position of the decimal point, Thus all such numbers which have the same digits as N (a given number), will be either divided or multiplied by any multiple of 10, and will be included in the group of $N \times 10^m$ where m is any + ve or − ve integer.

Let log N=I·abcd......where I is the characteristic and ·abcd......the mantissa,

then, $\log (N \times 10^m) = \log N + \log 10^m$ [from (1) § 9·2]
$= \log N + m \log 10$ [from (3) § 9·2]
$= I\,abcd\ldots + m$ [$\because \log_{10} 10 = 1$]
$= (1+\mathbf{m})\cdot\mathbf{abcd}\ldots\ldots$

We observe that ·abcd.. the mantissa remains the same, but the characteristic has changed from I to I+m with the change of number from N to $N \times 10^m$. I+m is positive or negative integer. Thus, if the position of the decimal point be changed or any number of zeros added to the right, the mantissa does not change.

Ex. 10. Given log 2345=3·3701, find the values of (*i*) log 23·45, (*ii*) log ·2345, (*iii*) log ·00002345, (*iv*) log 2345000

(*i*) $\log 23{\cdot}45 = \log 2345 \times 10^{-2} = \log 2345 + \log 10^{-2}$
$= 3{\cdot}3701 - 2 = 1{\cdot}3701.$

(*ii*) $\log {\cdot}2345 = \log 2345 \times 10^{-4} = \log 2345 + \log 10^{-4}$
$= 3{\cdot}3701 - 4 = 1{\cdot}3701.$

(iii) $\log {\cdot}00002345 = \log 2345 \times 10^{-8} = \log 2345 + \log 10^{-8}$
$= 3{\cdot}3701 - 8 = 5{\cdot}3701.$

(iv) $\log 2345000 = \log 2345 \times 10^3 = \log 2345 + \log 10^3$
$= 3{\cdot}3701 + 3 = 6{\cdot}3701.$

Note 1. —We observe, from the above example, that *mantissa is independent of the position of decimal point.* All numbers here have the same order of digits 2345, and have the same mantissa as ·3701, only the characteristic varying.

Note 2 :—The effect of multiplying a number by any integral power of 10 (+ve or negative) is to produce another number having the same order of digits by merely shifting the decimal point.

Ex. 11 Find the number of digits in $(\underline{|4})^2 \times 2^8$ given $\log 2 = {\cdot}30103$, $\log 3 = {\cdot}4771213$.

Sol. Let $x = (\underline{|4})^2 \times 2^8 = 4^2.3^2.2^2.1^2 \times 2^8,$
$= 3^2 \times 2^{14}.$

$\log x = 2 \log 3 + 14 \log 2,$
$= 2 \times {\cdot}4771213 + 14 \times {\cdot}30103$
$= {\cdot}9542426 + 4{\cdot}21442,$
$= 5{\cdot}1686626.$

$\therefore$ x or $(\underline{|4})^2 \times 2^8$ is a number the logarithm of which has 5 as characteristic. Hence, the number of digits in $x = 5 + 1 = 6$.

Note : [$\because$ Characteristic is one less than the number of digits.

$\therefore$ The number of digits is one more than the characteristic.]

EXERCISE XVIII

1. Given $\log 4{\cdot}317 = {\cdot}6352$, and $\log {\cdot}0127 = \bar{2}.1038$,

   find, by inspection, the values of :—

   (*i*) log 4317, (*ii*) log 12.7 (*iii*) log ·004317,
   (*iv*) log 127 (*v*) log ·4317, (*vi*) log ·127.
   (*vii*) log 127000

2. Given $\log 2 = {\cdot}30103$, $\log 3 = {\cdot}4771$, $\log 7 = {\cdot}8451$.
   Find the digits in the integral part of the following numbers :

   (*i*) $9^{27} \times 16^4$, (*ii*) $(12{\cdot}5)^{100}$, (*iii*) 5^{200}, (*iv*) $(980)^{42}$.

3. Find the number of zeros after the decimal point and before the first significant digit in the following :

(*i*) 3^{-25}, (*ii*) $(0081)^{100}$, (*iii*) $(00025)^{19}$, (*iv*) $\left(\frac{1}{2100}\right)$

10·8 Use of Four Figure Log Tables.

Tables for common logarithms are used when the numbers are not integral powers of 10, and they have a fractional part (mantissa) in the value of logarithm. When this mantissa extends only to four figures after the decimal point in any set of tables, they are called Four-figured Tables. An extract from four-figured Tables will explain points of the use of tables :-

USE OF FOUR FIGURE LOG TABLE

	0	1	2	3	4	5	6	7	8	9	123	456	789
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	112	233	455
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	112	233	455

10.8.1. When the mantissa of the logarithm of a number is required from the four-figured tables, we have to make the number *four-digited* by approximating to 4 digits if it has more than four digits, or by adding zeros on the right if it has less than four digits. We have to take four significant digits from the left irrespective of the decimal point. The *two digits* from the left of the given number will be found in the extreme left column in the Tables headed by a vacant square. The *third digit* from the left is to be taken from one of the ten columns headed by 0, 1, ...9 ; and the *fourth digit* of the given number from the left is to be found from one of the nine columns of Mean Differences, the small columns, on the right.

Ex. 12. From the extract above, find the value of (*i*) log 76·85, (*ii*) log ·0007507, (*iii*) log ·75385.

(*i*) Neglecting the decimal point the digits of the number are 7685. Now looking up for 76 in the extreme left column

and moving along the row across 76, we get 8854 in the column headed by 8 (third figure of the number). Moving further in the same row in the Difference Columns under 5 (the fourth digit), we get 3 which stands for ·0003. Hence mantissa is equal to ·8854 *plus* ·0003 or ·8857. The characteristic, by § 9·5, is evidently one. Thus log 76·85 = 1·8857.

(*ii*) Here the significant digits of the number are 7507. Proceeding along the horizontal row across 75, in the vertical column under 0, we get 8808 and the number in difference columns vertically under 7 is 4; the work may be arranged as

log ·000750 = $\bar{4}$·8808   from § 9·5

Difference for 7 = ·0004

log ·0007503 = $\bar{4}$·8812

(*iii*) Here ·75385 = ·7539 correct upto four places of decimal pt. Now proceeding along the horizontal row across 75, under vertical column headed by 3, we get 8768, and moving further in the Difference Columns under 9 we find 5. Thus

log ·753 = $\bar{1}$·8768

Diff. for 9 = ·0005

∴ log. 7539 = $\bar{1}$·8773

Thus, log ·75385 = $\bar{1}$·8773 approximately.

Note: We can find, from the Tables here, that log ·7538 and log ·7539 are the same as $\bar{1}$·8773. This means that the mantissas of the two numbers have no differences upto four places. In the subsequent figures, the difference will occur and the two values are never the same.

10.8.2. Antilogarithms: If the logarithm of a number is given, the number is called the *antilogarithm* of the *given quantity*. This is just the reverse of taking logarithms. Let $x = \log_a N$, then $N = \textbf{antilog}_a \mathbf{x}$. Thus, the operator '*antilog*' on any quantity, means number which is the result of raising the base to that quantity (the quantity being index of the power of base).

10.8.3. How to find Antilogarithms ?

Ex. 13. Find Antilog 3.8778, from antilog tables.

ANTILOG TABLE

	0	1	2	3	4	5	6	7	8	9	123	456	789
·87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	235	7910	[illegible]

Arrange the result thus,

antilog	No.	
·877	7·534	(a number in the column headed by 7 and in the row across ·87)
Diff. for .0008	13	(a number in Diff. Clms. headed by 8 and in the row across ·87).
.8778	7.547	

$\therefore$ antilog ·8778=7·547.

Hence, antilog $\bar{3}$·8778=.007547 (by Note (*ii*), above]

Ex. 14. Find the eleventh root of ·007547.

Sol. Set $x=(\cdot 007547)^{\frac{1}{11}}$,

or $\log x=\frac{1}{11}\log(\cdot 007547)$,

or $\log x=\frac{1}{11}(\bar{3}\cdot 8778)$,

$=\frac{1}{11}(\bar{11}+8\cdot 8778)$,

$=\bar{1}\cdot 8071$.

$\therefore$ $x=\text{antilog }\bar{1}\cdot 8071$,

$=.6413$ (From antilog tables.)

10.8.4. Tables of Logarithmic Trigonometric numbers.

In many problems, where trigonometric calculations are required, we come across with the logarithms of trigonometrical numbers. Expressions like *log Sine, log Cosine, log tan etc.* are usually required in solutions of triangles. One of the ways of finding the logarithms of trigonometrical numbers is first to find the Sine, Cosine, tangent *etc.* of the given angle from the *Tables of Natural Functions*, and then to find the logarithm of the obtained number from the Tables of Logarithms. For example to calculate log tan 52°, first find tan 52° = 1·2799 from tables of Natural tangents and then consult the log-tables to get log 1·2799 = ·1072. Thus, log tan 52° = ·1072.

To avoid the inconvenience of using *two tables*, separate tables giving *logarithms of trigonometrical numbers* have been calculated. An extract from four-figure Tables of logarithmic tangents is given below.

LOGARITHMS OF TANGENTS

Degrees	0′ 0°·1	6′ 0°·1	12′ 0°·2	18′ 0°·3	24′ 0°·4	30′ 0°·5	36′ 0°·6	42′ 0°·7	48′ 0°·8	54′ 0°·9	Mean Diff 1′ 2′ 3′	4′ 5′
44°	1·9848	9864	8879	9894	9909	9924	9939	9955	9970	9985	3 5 8	10 13
45°	·0000	0015	0030	0045	0061	0076	0091	0106	0121	0136	3 5 8	10 13

Ex. 15. Find log tan 44°32′.

log tan 44°30′ = 1·9924 [In the row across 44° and column headed by 30′]

Diff. for 2′ = .0005 [In the Diff. column under 2′]

log tan 44°32′ = 1·9929* add in the case of tangent.

*****Note.** : *With the increase of angles from 0 to 90°, the Difference is* **added** *in the case of* **logarithms of Sine, Secant and tangent,** *and* **subtracted in the** *case of* **logarithms of Cosine, Cosecant and Cotangent.**

Ex. 16. Find x, if $\log \tan x = .0128$.

The given number is not in the Tables; the one nearest to it and less than it is $\cdot 0121$ in the column under 48′ across 45°.

$\therefore \log \tan 45°48' = \cdot 0121$.

But $\cdot 0128 - \cdot 0121 = \cdot 0007$; this difference 7 is not found in the Difference Columns, while we find 5 under 2′ and 8 under 3′ in the Diff. Columns, 3′ is selected, as 7 is nearer to 8 than to 5, and is added in 45°48′.

$\therefore \log \tan 45°51' = \cdot 0128$

Hence $x = 45°51'$.

Note: The value in the above example could have been interpolated by the Principle of Proportional Parts but this principle is more suited to seven-figured tables rather than in four-figured tables where the above method gives sufficiently accurate results.

10.9. Tabular Logarithms.

As the Sine, Cosine of an angle is always less than one, the characteristics of their logarithms are always *negative*, which is also the case with logarithms of the tangent of angles less than 45° and Cotangent of angles greater than 45°; to avoid the inconvenience of printing the negative characteristics, in some cases, the values of logarithms are tabulated by adding 10 to the true values of log arithms of trigonometric numbers and such values are called *Tabular logarithms*. The symbol L. is used to denote these 'tabular logarithms'.

Thus, $\log \tan 44°32' = \bar{1}\cdot 9929$

$$\begin{aligned} \text{While L tan } 44°32' &= 10 + \log \tan 44°32' \\ &= 10 + \bar{1}\cdot 9229 \\ &= 9\cdot 9929 \end{aligned}$$

Note: To get the true value of logarithm of trigonometric number, the corresponding Tabular value must be diminished by 10.

Note 2. In terms of tabular logarithms, the extract of Ex. 10·8.4. will be read thus :

Tabular Logarithmic tangents

Degrees	0′	6′	12′	18′	24′	30′	36′	42′	48′	56′	Mean Diff.	
											1′ 2′ 3′	4′ 5′
44°	9·9848	9864	9879	9894	9909	9924	9939	9955	9970	9985	3 5 8	10 13
45°	10·0000	0016	0030	0045	0061	0076	0091	0106	0121	0136	3 5 8	10 13

10.10. Principle of Proportional Parts.

In case the angle contains integral number of degrees and minutes, the tabular logarithm is directly obtained from the tables ; but when the angle contains seconds also, the value of the logarithm is interpolated by the principle of proportional parts which states that *the increase in the logarithm of a number is proportional to the increase in the number itself.* When used in connection with logarithms of trigonometric numbers, we may state :

"The small differences between the angles are proportional to the corresponding differences between the logarithms of the trigonometrical numbers of those angles".

Ex. 17. Given L $\cos 34°44' = 9{\cdot}9147729$ and L $\cos 34°45' = 9{\cdot}9146852$.

find the value of L $\cos 34°44'27''$.

Here L $\cos 34°44' = 9{\cdot}9147729$

L $\cos 34°45' = 9{\cdot}9146852$

diff. for $1' = {\cdot}0000877$.

For an increase of 1′ or 60″ in the angle, there is *decrease* of ·0000877 in the logarithm, hence for an increase of 27″ in the angle, the corresponding *decrease* is $\frac{27}{60} \times {\cdot}0000877$ *i.e.* ·0000395.

Hence $L \cos 34°44' = 9.9147729$
$\text{Diff. for } 27'' = \cdot 0000395$ (Subtract)
$L \cos 34°44'27'' = 9\cdot 9147334$

Ex. 18. Find in degrees, minutes and seconds the angle whose Sine is $\cdot 6$, given that

$\log 6 = \cdot 7781513$, $L \sin 36°52' = 9\cdot 7781186$

$L \sin 36°53' = 9\cdot 7782870$.

Sol. Let x be the required angle.

$\log \text{Sin } x = \log \cdot 6 = \bar{1}\cdot 7781513$

$L \text{ Sin } x = 10 + \log \text{Sin } x = 9\cdot 7781513$

$L \text{ Sin } x = 9\cdot 7781513$ $\quad$ $L \text{ Sin } 36°53' = 9\cdot 7782870$

$L \text{ Sin } 36°52' = 9\cdot 7781186$ $\quad$ $L \text{ Sin } 36°52' = 9\cdot 7781186$

$\text{diff.} = 327$ $\quad$ $\text{diff. for } 1' = 1684$

Corresponding increase in the angle $= \dfrac{327 \times 60''}{1684} = 11\cdot 7''$

$\therefore$ $x = 36°52'\ 12''$ approximately.

```
1684) 19620 (11·7
      1684
      2780
      1684
      11960
```

Note : In the application of principle of proportional parts to the trigonometrical numbers, it is always to be kept in view that all the co-numbers decrease with the increase in the angle.

EXERCISE XIX

1. Show that $\log 2 = \log \dfrac{133}{65} + 2 \log \dfrac{13}{7} - \log \dfrac{143}{90} + \log \dfrac{77}{171}$

2. Solve the following Equations, given $\log 2 = \cdot 30103$, $\log 3 = \cdot 47712$, $\log 7 = \cdot 84510$:

(i) $2^{x} \cdot 3^{2x} = 5^{x-1}$.

(ii) $7^{3x+2} + 4^{x+2} = 7^{3x+1} + 4^{x+3}$.

(iii) $7^{2x} \div 2^{x-4} = 3^{3x-7}$.

3. Find the number of digits in 25^5.

4. Find the number of digits in 3^{43},

5. Given Sin $23°15' = \cdot3947439$, Sin $23°16' = \cdot3950111$, find Sin $23°15'20''$.

6. Given L Sin $23°15' = 9\cdot59530$, L Sin $23°16' = 9\cdot59658$, find L Sin $23°15'20''$.

7. Given L Cos $24°4' = 9\cdot9605048$,
L Cos $24°5' = 9.9604484$, find L Cos $24°4'32''$.

8. Given L Sin $14°6' = 9\cdot386704$, find L Cosec $14°6'$.
[**Hint** : Sin $\theta \times$ Cosec $\theta = 1$, $\therefore$ L Sin $\theta +$ L Cosec $\theta = 20$ etc.]

9. Find the angle x, where L Cot $x = 9\cdot5254782$,
given L Cot $71°27' = 9\cdot5257779$,
L Cot $71°28' = 9\cdot5253589$.

10. Find the time in which a pice will amount to a rupee if rate of interest being allowed 7% compound interest. Given log $2 = \cdot3010$, log $1\cdot07 = \cdot0294$.

CHAPTER XI

Solution of Triangles

11.1. Elements of a triangle.

Let ABC be a triangle. The capital letters A, B, C denote the angles and the small letters a, b, c represent respectively the sides opposite to these angles. Thus the three angles and three sides together make up the six fundamental elements of a triangle. We know from geometry that a triangle is *uniquely* drawn if we are given :

(*i*) **One side and any two angles, the sum of the given angles being less than 180°.**

(*ii*) **Two sides and the included angle.**

(*iii*) **Two sides and the angle opposite to one of them. (There may be ambiguity if the given angle is opposite to smaller of the given sides.)**

(*iv*) **Three sides, every side being less than the sum of the other two.**

Thus, we observe that *necessary data* to determine the shape and size of a triangle (except in one case when only three angles are known) is that *three of its elements* must be known out of which *one must be a side*. This process of calculating the unknown three elements from the given three elements of a triangle is called **Solution of triangles.**

11.2. Solution of a right angled triangle.

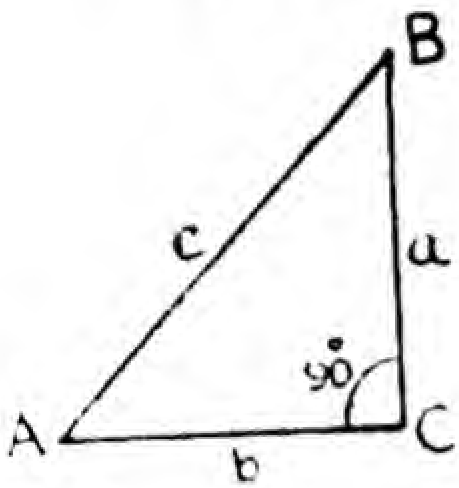
In a rt. ∠ed △, the right angle is always known out of the three elements which determine the triangle completely. Out of these other two elements. for solution, at least one must always be a side. A rt. angle, thus, can be completely solved under the following cases :

(*i*) **The two adjacent sides other than the hypotenuse be known.**

(*ii*) **The hypotenuse and any other side be known.**

(*iii*) **The hypotenuse and any other angle be known.**

(*iv*) **Any angle and any side other than the hypotenuse be known.**



Let ABC be a right angled triangle right angled at C, c will be the hypotenuse, a and b, the two adjacent sides.

Case (*i*) Given a, and b

To find A, $\tan A = \frac{a}{b}$ (1)

For logarithmic calculations, $L \tan A = 10 + \log a - \log b$.

To find B, $90° - A = B$ (2)

To find C, $\frac{c}{a} = \text{Cosec } A$ or $C = a \text{ Cosec } A$ (3)

For logarithmic calculations, $\log c = \log a + 10 - L \sin A$.

Hence (1), (2) & (3) determine all the three unknown elements A, B, c.

Note : The hypotenuse c is also given by $c = \sqrt{a^2 + b^2}$ but this relation is not suitable for logarithmic work.

Case (ii). Given c and a.

To find **A**, $\frac{a}{c} = \sin A$. (1)

For logarithmic calcutations, $L \sin A = \log a - \log c + 10$

To find B, (a) $90° - A = B$ (2)

To find b,
$$\left.\begin{array}{ll} (a) & b = \sqrt{(c+a)\ (c-a)} \\ (b) & b = c \text{ Cos } A \\ (c) & b = a \text{ Cot } B \end{array}\right\} \quad (3)$$

In all the above relations, logarithmic calculations can be adopted.

Case (*iii*) Given c and A.

To find B, $90° - A = B$ (1)

To find a, $a = c$ Sin A (2)

To find b, $b = c$ Sin B or c Cos A (3)

Case (*iv*) Given A, a

To find B, $B = 90° - A$ (1)

To find b, $b = a$ Cot A (2)

To find c, $c = a$ Cosec A (3)

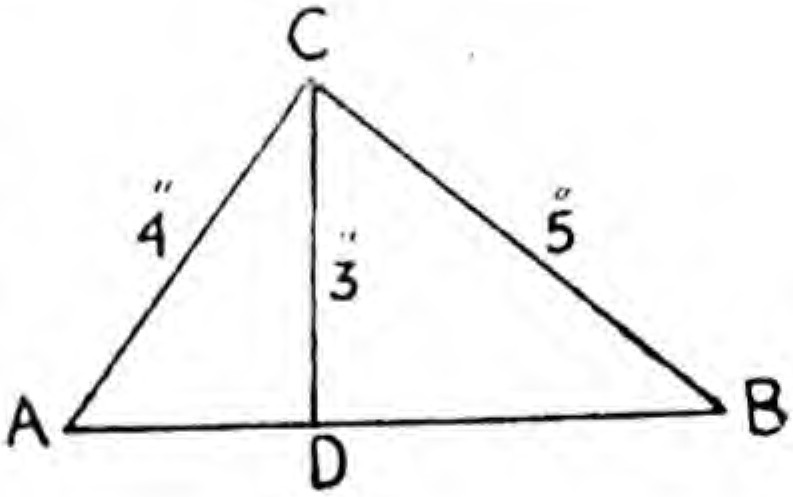
For logarithmic calculations, (2) and (3) become

$\log b = \log a - \text{L} \tan A + 10$, and

$\log c = \log a - \text{L Sin } A + 10$, respectively.

Ex. 1. The length of the perpendicular from one angle of a triangle upon the base is 3 inches, and the lengths of the sides containing this angle are 4 and 5 inches. Find the angles, having given log 2 = .30103, log 3 = .4771213, L Sin 36°52′ = 9.7781186; diff. for 1′ = 1684, L Sin 48°35′ = 9.8750142, diff. for 1′ = 1115.

Now,



$$\text{Sin } A = \frac{3}{4}$$

$$\therefore \text{L Sin } A = 10 + \log 3 - 2 \log 2$$
$$= 10 + 4771213 - 60206$$

Here, L Sin A = 9.8750613

$$\frac{\text{L Sin } 48°35' = 9.8750142}{\text{diff.} \quad = \quad 471}$$ diff. for 1′ = 1115

$$= \frac{471}{1115} \times 60''$$
$$= 25'' \text{ nearly.}$$

$\therefore \quad A = 48°35'25''$.

```
1115) 28260 (25
      2230
      ----
       5960
       5575
       ----
        385
```

Also, $\text{Sin } B = \frac{3}{5} = \frac{3 \times 2}{10}$

or $L \text{ Sin } B = 10 + \log 3 + \log 2 - \log 10$
$= 9 + .4771213 + .30103 \doteq 9.7781513$

$\therefore$ $L \text{ Sin } B = 9.7781513$
$L \text{ Sin } 36°52' = 9.7781186$ diff. for $1' = 1684$
diff. = 327

$\therefore$ diff. $= \frac{327 \times 60''}{1684} = 11{\cdot}6''$
$= 12''$ nearly
Hence, $B = 36°52'12''$.

```
1684) 19620 (11·6
      1684
      2780
      1684
      10960
```

$\therefore$ $C = 180° - 48°35'25'' - 36°52'12''$
$= 94°32'23''$.

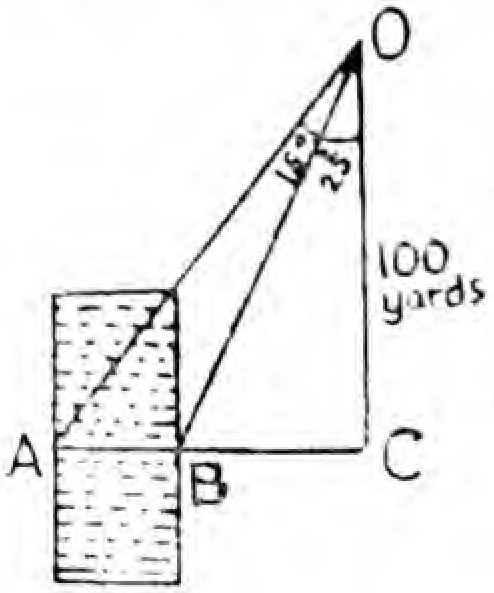
Ex. 2. To determine the breadth AB of a canal, an observer places himself at C in the straight line AB produced through B and then walks 100 yds. at right angles to this line. He then finds that AB and BC subtend angles 15° and 25° at his eyes. Find the breadth of the canal, given

$L \text{ Cos } 25° = 9{\cdot}9572757$, $L \text{ Cos } 40° = 9{\cdot}8842540$,

$L \text{ Cos } 75° = 9{\cdot}4129962$, $\log 37279 = 4{\cdot}5714643$,

$\log 3728 = 3{\cdot}5714759$.

Let AB be the canal and O the new position of the observer. Draw AD perp. from A on OB produced, then



$\widehat{OAD} = 75°$, and $\widehat{BAD} = 25°$.
$\therefore$ In the rt. $\angle$ed $\triangle$ ABD,
AB Cos 25° = AD.
Again, in the rt. $\angle$ed $\triangle$OAD,
AD = OA Cos 75°
Also, in the rt. $\angle$ed $\triangle$ AOC,
OA Cos 40° = 100

$\therefore$ $AB \text{ Cos } 25° = \frac{100 \text{ Cos } 75°}{\text{Cos } 40°}$

Taking tabular logarithms, we get

$$\log AB + L\ Cos\ 25^\circ - 10 = \log 100 + L\ Cos\ 75^\circ - L\ Cos\ 40^\circ$$

or $\log AB = 10 + 2 + 9{\cdot}4129962 - 9{\cdot}8842540 - 9{\cdot}9572757$

or $\log AB = 21{\cdot}4129962 - 19{\cdot}8415297$
$= 1{\cdot}5714665.$

Now, $\log 37{\cdot}280 = 1{\cdot}5714759$
$\log 37{\cdot}279 = 1{\cdot}5714643$
Diff. for $\cdot 001 = \cdot 0000116$

$\log AB = 1{\cdot}5714665$
$\log 37{\cdot}279 = 1{\cdot}5714643$
Diff. $= \cdot 0000022$

$$\therefore \quad \text{Diff.} = \frac{\cdot 0000022 \times \cdot 001}{\cdot 0000116} = \cdot 00019 \text{ nearly.}$$

$\therefore \quad AB = 37{\cdot}279 + \cdot 00019 = 37{\cdot}27919$ yds.

EXERCISE XX

1. Solve the triangle ABC, where $C = 90^\circ$, $a = 50$, $B = 75^\circ$.
2. If $a = 30$, $b = 300$, find A in order that B may be a rt. angle, having given L Sin $5^\circ 44' = 8{\cdot}9995595$,
   diff. for $1' = \cdot 0012565$.
3. In a $\triangle$, $a = 384$, $b = 330$, $C = 90^\circ$, find other angles, given $\log 11 = 1{\cdot}0413927$, $\log 20 = 1{\cdot}3010300$,
   L tan $49^\circ 19' = 10{\cdot}0656886$, L tan $49^\circ 20' = 10{\cdot}0659441$.
4. A tower 150 ft. high throws a shadow 75 ft. long upon the horizontal plane upon which it stands. Find the sun's altitude, having given $\log 2 = \cdot 30103$, L tan $63^\circ 26'$ $= 10{\cdot}3009994$, L tan $63^\circ 27' = 10{\cdot}3013153$.
5. Solve the triangle of which two sides are equal to 10 and 20, and of which the included angle is 90°; given $\log 2 = \cdot 30103$, and L tan $26^\circ 33' = 9{\cdot}6986847$.
   diff. for $1' = 3160$.

11. 4. Solution of oblique Angled Triangles.

Case I. Given two angles and one side, to solve the triangle.

Let a, A and B be given, then the third angle $C = 180^\circ - (A + B)$

For the rest we can apply the *Sine Formula*

$$\therefore \quad b=\frac{a \text{ Sin B}}{\text{Sin A}},\ c=\frac{a \text{ Sin C}}{\text{Sin A}}$$

$$\left[\because \frac{a}{\text{Sin A}}=\frac{b}{\text{Sin B}}=\frac{c}{\text{Sin C}}\right]$$

Ex. 3. In a $\triangle$ ABC, $A=72^\circ\ 43'$, $B=64^\circ\ 23'$ and $c=473$. Find C, b, and a

Sol. $C=180^\circ-(A+B)=180^\circ-(72^\circ\ 43'+64^\circ\ 23')$
$=180^\circ-(137^\circ\ 6')=42^\circ\ 54'$

Now by Sine Formula, $\frac{a}{\text{Sin A}}=\frac{c}{\text{Sin C}}$

$$\therefore a=\frac{c \text{ Sin A}}{\text{Sin C}}$$

$$\therefore \log a=\log\frac{c \text{ Sin A}}{\text{Sin C}}=\log c+\log \text{Sin A}-\log \text{Sin C}$$

$$=\log 473+\log 72^\circ\ 43'-\log \text{Sin } 42^\circ\ 54'$$

$$=2.6749+\bar{1}.9799-\bar{1}.8330$$

$$=2.8218=\log 663.4$$

$$\therefore \quad a=663.4$$

Again by Sin formula, $\frac{b}{\text{Sin B}}=\frac{c}{\text{Sin C}}$

$$\therefore \quad b=\frac{c \text{ Sin B}}{\text{Sin C}}$$

$$\therefore \log b=\log\frac{c \text{ Sin B}}{\text{Sin C}}=\log c+\log \text{Sin B}-\log \text{SinC}$$

$$=\log 473+\log \text{Sin } 64^\circ\ 23'-\log \text{Sin } 42^\circ\ 54'$$

$$=2.6749+\bar{1}.9553-\bar{1}.8330=2.7972$$

$$=\log 626.9$$

$$\therefore \quad b=626.9$$

EXERCISE XXI

Solve the triangle, given :—

1. $B=88° 36'$; $C=31° 55'$ $a=53$
2. $B=64° 23'$; $C=72° 43'$ $a=18.92$

   Find b and c

3. $a=226.9$; $B=73° 55'$; $C=39° 45'$
4. $B=64° 23'$; $C=72° 43'$; $a=18.9$
5. $A=66° 38'$; $B=26° 14'$; $c=32.42$

11.5. Case II Given three sides a, b, c of a triangle to solve the triangle.

Since the sides are known, the semiperimeter s, and the quantities $s-a$, $s-b$, and $s-c$ can be found out easily. Also

$$\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

$$\therefore \quad \log\tan\frac{A}{2}=\tfrac{1}{2}\left[\log\ (s-b)+\log\ (s-c)-\log\ s-\log\ (s-a)\right]$$

This will give us $\frac{A}{2}$. Doubling this, we can get A.

Similarly, we can get from the formula

$$\tan\frac{B}{2}=\sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$$

and C can lastly be got from the equation

$$c=180°-(A+B)$$

Ex. 4. Solve the triangle whose sides

$a=32,\ b=40,\ c=66$

(P.U. 1946)

Sol. $2s=32+40+66=138 \therefore s=69$

$s-a=37$, $s-b=29$, $s-c=3$

Now $\log \tan \frac{A}{2} = \log\sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \log\sqrt{\frac{29\times 3}{69\times 37}}$

$= \frac{1}{2}[\log 29 + \log 3 - \log 69 - \log 37]$

$= \frac{1}{2}[1{\cdot}4624 + {\cdot}4771 - 1{\cdot}8388 - 1{\cdot}5682]$

$= \frac{1}{2}[1{\cdot}9395 - 3{\cdot}4070]$

$= \frac{1}{2}[-2 + 3{\cdot}9395 - 3{\cdot}4070]$

(Please note this step)

$= 1 + {\cdot}2662 = 1{\cdot}2662 = \log \tan 10^\circ\ 28'$

$\therefore\ \frac{A}{2} = 10^\circ\ 28' \quad \text{or} \quad A = 20^\circ\ 56'$

Similarly, $\log \tan \frac{B}{2} = \log\sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$

$= \log\sqrt{\frac{3\times 37}{69\times 29}}$

$= \frac{1}{2}[\log 3 + \log 37 - \log 69 - \log 29]$

$= \frac{1}{2}[{\cdot}4771 + 1{\cdot}5682 - 1{\cdot}8388 - 1{\cdot}4624]$

$= \frac{1}{2}[2{\cdot}0453 - 3{\cdot}3012]$

$= \frac{1}{2}[-2 + {\cdot}7441] = \log \tan 13^\circ\ 15'$

$\therefore\ B = 26^\circ\ 30'$

and $C = 180^\circ - (A+B) = 180^\circ - (47^\circ\ 26')$
$= 132^\circ 34'$

EXERCISE XXII

Solve the triangle, if

1. $a=31$, $b=42$, $c=57$
2. $a=4584$, $b=5140$, $c=3624$, find A
3. $a=8$, $b=9$, $c=10$
4. $a=32$, $b=40$, $c=66$
5. $a=229{\cdot}2$, $b=181{\cdot}2$, $c=257$

6. Find the area of the $\triangle ABC$, the radius of the in-circle, and solve it when

   $a=725$ ft., $b=548$ ft., $c=474$ ft., given

   log 8735=3·94126, log 1485=3·17173

   log 3255=3·51255 ; log 3995=3·601

   log 5·513=·11353 ; log 1·4868=·17227

   L tan 45° 2′=10·00046 ; L tan 24° 33′=9·65972

11.6 Case III. when two sides and the included angle are given, to solve the triangle

Let the given sides be a and b $(a>b)$ and C the included angle. Now from the formula

$$\tan\frac{A-B}{2}=\frac{a-b}{a+b}\,\text{Cot}\,\frac{C}{2} \quad \text{(Napier's Analogy)}$$

$$=\frac{a-b}{a+b}\tan\left[\frac{\pi}{2}-\frac{C}{2}\right]$$

we get (by taking logarithms)

$$\log\tan\frac{A-B}{2}=\log(a-b)-\log(a+b)+\log\tan\left(\frac{\pi}{2}-\frac{C}{2}\right)$$

From this we can get $\dfrac{A-B}{2}$ (i)

Also $\dfrac{A+B}{2}=90^\circ-\dfrac{C}{2}$ (ii)

$\therefore$ from (*i*), (*ii*) we can get A and B

Lastly, the side a can be found from the formula

$$\frac{a}{\text{Sin A}}=\frac{b}{\text{Sin B}}$$

This will give us $a=\dfrac{b\,\text{Sin A}}{\text{Sin B}}$

Hence $\log a=\log b=\log \text{Sin A}-\log \text{Sin B}$

Ex. 5. Solve the triangle, when :—

$b=237,\ c=158,\ A=66^\circ\ 40'$

Sol. $b+c=395,\ b-c=79,\ B+C=180^\circ-66^\circ\ 40'$
$=113^\circ\ 20'$

Also $\tan\dfrac{B-C}{2}=\dfrac{b-c}{b+c}\operatorname{Cot}\dfrac{A}{2}=\dfrac{b-c}{b+c}\tan\dfrac{B+C}{2}$

$$=\frac{79}{395}\tan 56^\circ\ 40'$$

$$=\tfrac{1}{5}\tan 56^\circ\ 40'$$

$$\therefore \log\frac{B-C}{2}=\log\left[\tfrac{1}{5}\tan 56^\circ\ 40'\right]$$

$$=\log 56^\circ\ 40'-\log 5$$

$$=\cdot1820-\cdot6990$$

$$=-\cdot5170=\bar{1}\cdot4830$$

$$\therefore \frac{B-C}{2}=16^\circ\ 55'$$

or $B-C=33^\circ\ 50'$

Also $B+C=113^\circ\ 20'$

$\therefore \qquad B=73^\circ\ 55'$ and $C=39^\circ\ 45'$

Also by **Sine Formula** $a=\dfrac{c\operatorname{Sin}A}{\operatorname{Sin}C}$

$$\therefore \log a=\log c+\log\operatorname{Sin}A-\log\operatorname{Sin}C$$

$$=\log 158+\log\operatorname{Sin}66^\circ\ 40'$$
$$-\log\operatorname{Sin}39^\circ\ 45'$$

$$=2\cdot19866+\bar{1}\cdot96294-\bar{1}.80581$$

$$=2\cdot35579=\log(226\cdot9)$$

$\therefore\ a=226\cdot9.$

Ex. 6. $a=9,\ b=7,\ C=47^\circ\ 25'$, find other angles it being given that $\log 2=\cdot3010300$

L tan $15^\circ-53'=9\cdot4541479$
L tan $66^\circ-17'-30''=10\cdot3573942$
Difference for $1'=\cdot0004797$.

Sol. $C=47° \; 25'$ $\therefore$ $A+B=180°-(47° \; 25')$

or $\frac{A+B}{2}=66° \; 17' \; 30''$ ......(i)

Now $\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2}=\frac{a-b}{a+b}\tan\frac{A+B}{2}$

$=\frac{1}{8}\tan 66° \; 17' \; 30''$

Taking tabular logarithms, we get

$L\tan\frac{A-B}{2}=L\tan 66° \; 17' \; 30''-3\log 2$

$=10\cdot3573942-3\times\cdot3010300$

$=10\cdot3573942-\cdot9030900$

$=9\cdot4543042$

$L\tan 15°-53'=9\cdot4541479$

Diff $=1563$

Now diff. for $60''=4797$

$\therefore$ Diff. $=\frac{1563\times60''}{4797}=19''\cdot5$

$\therefore$ $L\tan\frac{A-B}{2}=L\tan 15°-53'-19''\cdot5$

$\therefore$ $\frac{A-B}{2}=15°-53'-19\cdot5''$ ...(ii)

From (i) and (ii) we get

$A=82°-10'-49\cdot5''$

$B=50°-24'+10\cdot5''$

EXERCISE XXIII

Solve the triangle, if

1. $a=21\cdot35$, $b=35\cdot21$, $\angle C=50°—48'$ find B *(P.U. 1954)*

2. $b=25\cdot1$, $c=14\cdot7$ and $A=47°$ *(P.U. 1948)*

3. $b=237$, $c=158$, $A=66°-40'$

4. $b=11, c=9, A=32°-30'$

5. $b=37{\cdot}2, c=22{\cdot}3, A=29°-38'$

6. If $b=14$, $c=11$, $A=60°$, find the remaining angles of the $\triangle ABC$, it being given that

   $\log 2=\cdot 30103$, $\log 3=\cdot 47712$

   $L \tan 11°-44'-29''=9{\cdot}31774$

7. If $b=27$, $c=23$, $A=44°-30'$ find B and C, having been given that $\log 2=\cdot 30103$, $L \cot 22°-15'=10{\cdot}3881591$, $L \tan 11°-3'=9{\cdot}2906713$ diff. for $1'=\cdot 0006711$

8. In a $\triangle ABC$, $c=1400$, $b=1300$ and $A=60°$, find B and C, given $\log 3=\cdot 4771213$, $L \tan 3°-40'=8{\cdot}8067422$.

11·7 Case IV. Given two sides and the angle opposite to one of them (Ambiguous Case may arise).

Let a, c and A are given. We also suppose that A is not a right angle (§ 10·3).

Law of Sines is the only formula required for solution. We know $\dfrac{b}{\sin B}=\dfrac{a}{\sin A}=\dfrac{c}{\sin C}=2R$, hence

(*a*) First find R from $\log 2R=\log a-L \sin A+10$.

(*b*) Find C from $\sin C=\dfrac{c}{2R}$.

(*c*) Find $B=180°-(A+C)$, and $b=2R \sin B$.

If the $\triangle$ is real *i.e.* the datas given are consistent, we observe that *c must be less than or equal to 2R* as no side of a triangle *can be greater than* the diameter of its circumcircle. Hence, at the time of finding C, the following possibilities may arise :—

(*i*) $\dfrac{c}{2R}=1$ or $\sin C=1$, *i. e.* C is definite—a rt. angle.

(*ii*) $\dfrac{c}{2R}<1$ whence $\sin C=\dfrac{c}{2R}$ leads to two possible values of C, one in the 1st quadrant (acute), and another in the 2nd quadrant (obtuse) ; but (*a*) if $a \geqslant c$ or $A \geqslant C$, C

cannot be obtuse, *it must be acute* since there cannot be two obtuse angles in a $\triangle$; (*b*) if, on the other hand, $a<c$ so that $A<C$, the given value A should necessarily be acute, (for, when C is acute, A must be acute, and if C is obtuse A cannot be obtuse because of non-occurence of two obtuse angles in a triangle). Thus both the values of C given by (*ii*) above are possible, C may be acute and greater than A, or obtuse. In case, therefore, the *given angle A is opposite to the smaller of the given sides* ($a<c$), *there may be ambiguity*, (unless, as in (*i*), the angle opposite to the larger of the sides is a rt. angle) and two different triangles can be found to satisfy the given data ; this case is called **Ambiguous Case** of the solution of triangles.

10. 8. Discussion of Ambiguous Case, another way trigonometrically.

Sine formula also gives $\quad \text{Sin } C=\dfrac{c \text{ Sin } A}{a}$.

(*i*) If $c \text{ Sin } A>a$, we have $\text{Sin } C>1$ which is impossible and there is *no triangle* with the given elements.

(*ii*) If $c \text{ Sin } A=a$, then $\text{Sin } C=1 \therefore C=90°$ Hence, if $A<90°$ (acute), there is *one triangle*, (Rt. angled) ; if $A>90°$ (obtuse), there is *no triangle*, since $A+B+C=180°$, the value of $C=90°$ is inadmissible.

(*iii*) If $c \text{ Sin } A<a$, $\text{Sin } C<1$ and there may be two real values of C, one acute and the other obtuse which are supplementary. These values may not always hold together, for,

(*a*) If $c<a$, then $C<A$, hence C must be *acute* whether A be acute or obtuse, for, in the latter case, two obtuse angles in a triangle are never possible and thus there is *only one triangle*.

(*b*) If $c=a$, then $\text{Sin } C=\text{Sin } A$. $C=A$ or $180°-A$, the latter value is not possible since $A+B+C=180°$, and the former value is admissible only if $A<90$; thus, if A is *acute*, there is *one triangle* (Isosceles), and if A is *obtuse*, there is *no triangle* ;

(*c*) if $c>a$ then $C>A$; and if also A is *obtuse* ($A>90°$), there can be *no triangle* as there cannot be two obtuse angles in a triangle ; if $A<90°$ (*acute*) both values of C are admissible corresponding to which there will be two values of B and hence also two values of b, since

$B=180°-(A+C)$ and $b=\dfrac{a \text{ Sin B}}{\text{Sin A}}$; there are, therefore, *two triangles* satisfying the above conditions.

Hence, *for the ambiguous case, the given angle should be acute and the side opposite to the given angle be less than the other side under* (*iii*) [*i.e.* $c \text{ Sin } A<a<c$]

To sum up

$c \text{ Sin A} > a$	*no solution.*
$c \sin A=a,\ A<90°.$	*one solution.*
$A>90°,$	*no solution*
$c \text{ Sin } A<a,\ c<a,$	*one solution.*
$c=a,\ A<90°,$	*one solution.*
$c=a,\ A>90°,$	*no solution.*
$c>a,\ A>90°,$	*no solution.*
$c>a,\ A<90°,$	*two solutions.*

10.9 Treatment of the ambiguous case geometrically.

Let us show geometrically how the ambiguity arises. We are given the elements (a, c, A) as before.

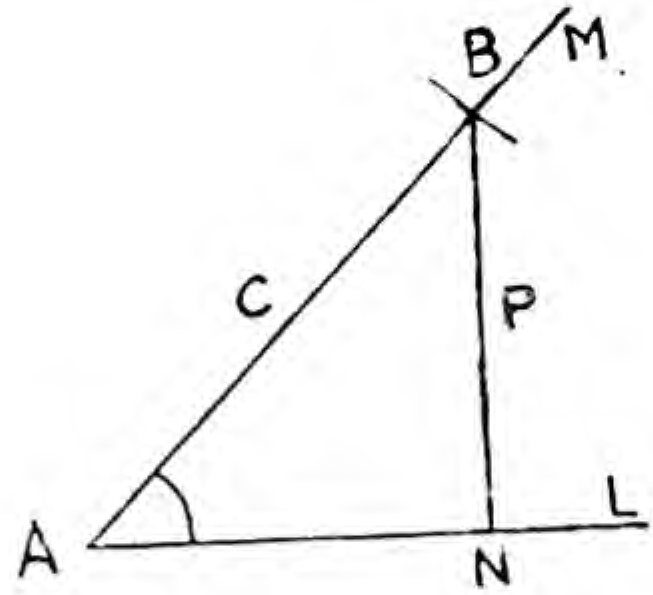
(A) Let A be *acute*.

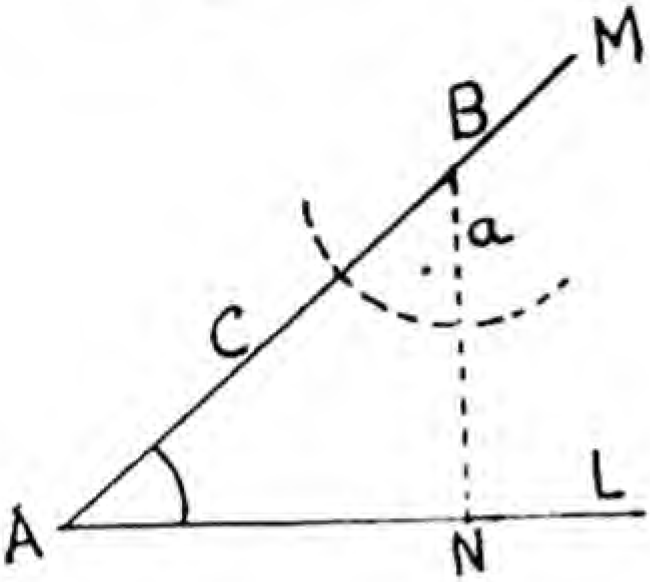
Construct $\widehat{LAM}$ on a line AL equal to the given angle. From AM cut off AB equal to the given side c. Draw BN perp. to AL say (p).

Then, $\text{Sin A} = \frac{BN}{c}$,

or $BN = c \text{ Sin A} = p$.

To find the position of the third vertex C, describe a circle with centre B and radius equal to a. It will meet AL for consistency. The following cases may arise according as :





(i) The circle meets AL in

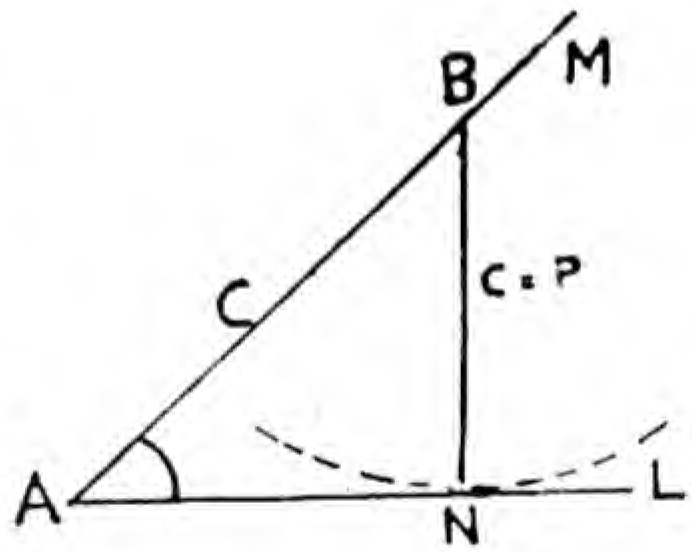
no points *i.e.* $a < p$.

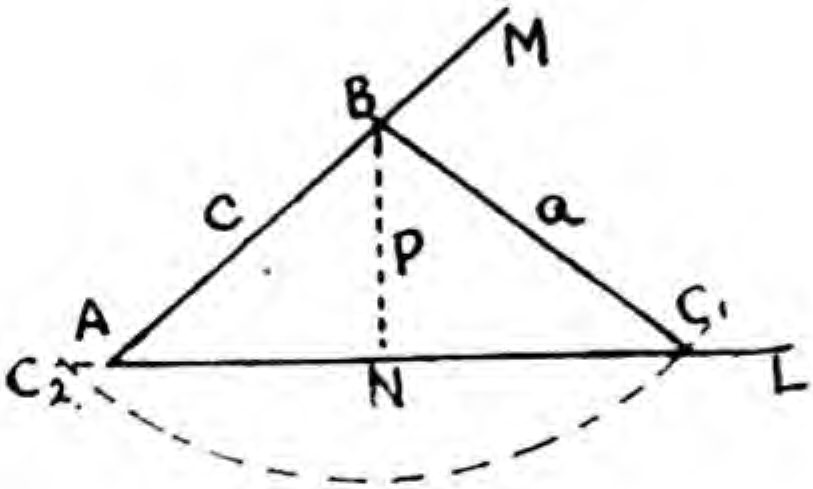
No real triangle.

(ii) The circle touches AL at

N *i.e.* $a = BN = p$.

One Right angled triangle.

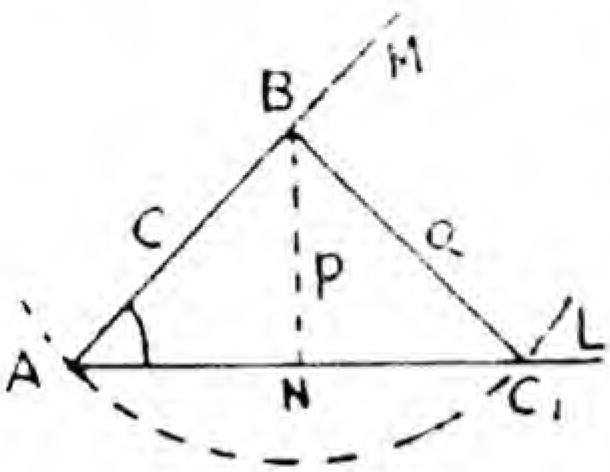




(iii) The circle cuts AL in two points *i.e.* $a > p$. or $c \text{ Sin A} < a$, three sub-cases arise according as :—

(a) The circle cuts AL in two points C_1 and C_2 which

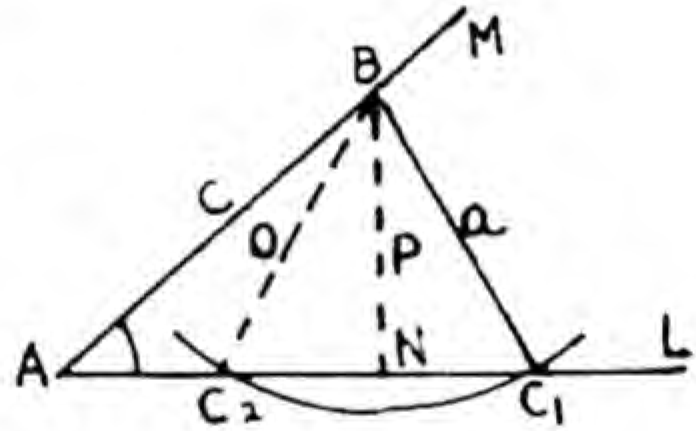
be on the opposite sides of A (one on the side of A as L and the other opposite to it). Here $a > p$, and also $> c$. Hence, *only one triangle* with the given data (ABC_1).



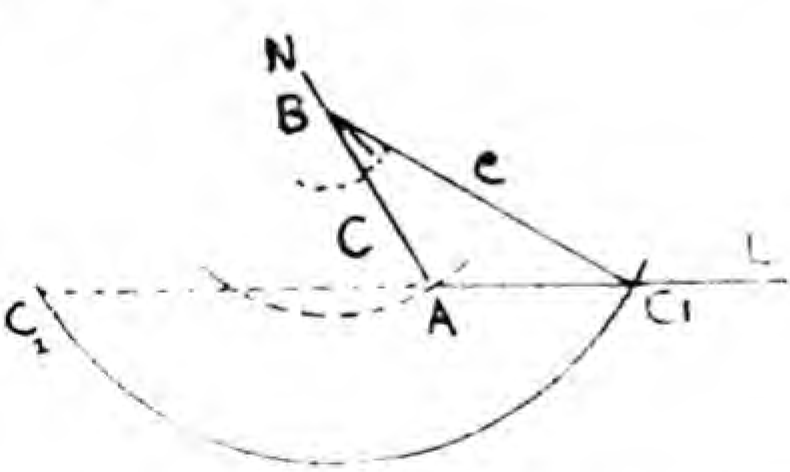
(*b*) The circle cuts AL in two points in such a way that one of the pts. C_2 coincides with A. Here, $p < a$ and $a = c$

Only one isosceles triangle.

(*c*) The circle cuts AL in two pts. such that both the points C_1 and C_2 be on the same side of A as L. Here, $p < a$ and $a < c$, both the $\triangle s$ ABC_1 and ABC_2 satisfy the given data ;



(C_2 being equal to $180° - C_1$). *Two triangles* possible with the given data.



(B) Let A be *obtuse.*

We observe that no triangle is possible satisfying the data, if $c \geqslant a$; and only one triangle is possible if $c \angle a$. *One triangle* ABC_1 only.

10.10 Algebraic treatment of the ambiguous Case.

Given a, c, A, we have from the Cosine formula (§ 8.7)

$$a^2 = b^2 + c^2 - 2bc \cos A$$

or $b^2 - 2bc \cos A - c^2 - a^2 = 0$ a quadratic equation.

Solving for b, we get

$$b=\frac{2c\,\text{Cos A}\pm\sqrt{4c^2\text{Cos}^2\text{A}-4(c^2-a^2)}}{2}$$

$$=c\cos A\pm\sqrt{a^2-c^2\text{Sin}^2A}.$$

Thus b *may* have *two real values* for the given data,

say $b_1=c\cos A+\sqrt{a^2-c^2\sin^2 A}$, and

$$b_2=c\,\text{Cos A}-\sqrt{a^2-c^2\,\text{Sin}^2A}.$$

1. Let A be acute, then we observe as follows :—

(*a*) If $a<c$ Sin A, or $a^2-c^2\ \text{Sin}^2A<0$, the two values of b are *imaginary* and hence *no real triangle* with given data.

(*b*) If $a=c$ Sin A, or $a^2-c^2\ \text{Sin}^2A=0$, the values of b are equal and real ($b_1=b_2=c$ Cos A), Hence the triangles are coincident and there is only *one triangle* satisfying the data.

(*c*) If $a>c$ Sin A or a^2-c^2 Sin^2 A>0, the two values of b will be real and *two real triangles* satisfying the given data will be *possible* only if b_1 and b_2 both are positive, for in case, b_1 or b_2 is negative the triangles formed by them will contain not the given angle A but its suplementary $180°-A$.

Thus for the **ambiguous case**, apart from $a>c$ Sin A,

$$c\,\text{Cos A}\pm\sqrt{a^2-c^2\,\text{Sin}^2\,A}>0$$

or $c^2\,\text{Cos}^2\,A>a^2-c^2\,\text{Sin}^2\,A$,

or $c^2>a^2$ *i.e.* $c>a$.

If (*i*) c Cos A$=\sqrt{a^2-c^2\,\text{Sin}^2\,A}$ or $c=a$, Only the value b_1 of b will be available and the other *i. e.* $b_2=0$, hence only *one triangle* is given with the given data (an Isosceles $\triangle$).

(*ii*) c Cos A$<\sqrt{a^2-c^2\,\text{Sin}^2\,A}$ or $c<a$.

Only *one triangle* satisfying the data will be possible as b_2 will become negative.

II. Let A be *obtuse*, c Cos A will be negative, and hence $b_2=c \text{ Cos A}-\sqrt{a^2-c^2 \text{ Sin}^2 \text{ A}}$ will always be negative, thus, if the other value b_1 is positive, there is only *one triangle possible.*

Now $b_1=c \text{ Cos A}+\sqrt{a^2-c^2 \text{ Sin}^2\text{A}}>0$

if $\sqrt{a^2-c^2 \text{ Sin}^2 \text{ A}}>-c \text{ Cos A}$

or $a^2-c^2 \text{ Sin}^2 \text{ A}>c^2 \text{ Cos}^2 \text{ A}$

or $a^2>c^2$ *ie.*, $a>c$.

In case $a=c$, $b_2=0$ and $b_1=2c$ Cos A, which is *negative* hence *no triangle*.

In case $a<c$, $c \text{ Cos A}+\sqrt{a^2-c^2 \text{ Sin}^2 \text{ A}}<0$, hence both the values of b are negative and *no triangle* can be possible.

EXERCISE

1. Discuss the ambiguities in the solution of triangles. (*Patna 1951*)

2. If $\log c-10=\log b-\text{L Sin B}$, test the ambiguity of the △.

3. Test the ambiguity of the triangle

   if $10+\log a>\log c+\text{L Sin A}$.

4. Show the following in the ambiguous case when a, c and A are given, and $c>a>c$ Sin A ;

   (*i*) $b_1+b_2=2\,c \text{ Cos A}$; (*ii*) $b_1\, b_2=c^2-a^2$

   where b_1 and b_2 are two values of b.

   [**Hint** : use § 10·10]

5. If a, b, A are given, and if c_1 and c_2 are the values of the third side, prove that

   (*i*) $c_1-c_2=2\sqrt{a^2-b^2 \text{ Sin}^2 \text{ A}}$

   (*ii*) $\text{Cos}\dfrac{\text{C}_1-\text{C}_2}{2}=\dfrac{b \text{ Sin A}}{a}$. [*Alld. 1941*]

   (*iii*) $c_1-c_2=2a \text{ Cos B}$.

[**Hint** : Solve $a^2=b^2+c^2-2bc$ Cos A for c, C_1-C_2 is the vertical angle of the Isos. △ B_1CB_2]

6. If b, c, and B of a $\triangle$ are given, and if a_1, a_2 are two values of the third side in the two solutions, A_1 and A_2 being the corresponding opposite angles, prove that

   (*i*) $a_1^2+a_2^2-2a_1a_2 \text{ Cos } 2B=4b^2 \text{ Cos}^2 B$;

   [**Hint** : $a_1+a_2=2c \text{ Cos } B$, $a_1 a_2=c^2-b^2$]

   (*ii*) $\dfrac{(b+a)^2}{1+\text{Cos A}}+\dfrac{(b-c)^2}{1-\text{Cos} A}=\dfrac{2c^2}{\text{Sin}^2 C}$ [*Banaras, 1942*]

   $\left[\textbf{Hint} : \text{start with } \tan^2 \dfrac{B-C}{2}=\dfrac{(b-c)^2}{(b+c)^2}\text{Cot}^2 \dfrac{A}{2}\right]$

7. If c_1 and c_2 be the values of the third side and B_1, C_1 and B_2, C_2 be the other two angles of the two triangles in an ambiguous case, then

   (*i*) $(c_1-c_2)^2+(c_1+c_2)^2 \tan^2 A=4a^2$,

   (*ii*) $\dfrac{\text{Sin } C_1}{\text{Sin } B_1}+\dfrac{\text{Sin } C_2}{\text{Sin } B_2}=2 \text{ Cos A}$

   [**Hint** : use $\dfrac{\text{Sin } C_1}{\text{Sin } B_1}=\dfrac{c_1}{b}$, $\dfrac{\text{Sin } C_2}{\text{Sin } B_2}=\dfrac{c_2}{b}$,

   $c_1+c_2=2b \text{ Cos A}$ etc.]

Ex. 13. Point out, giving reasons, the number of solutions in the triangles having the following data :—

(*i*) $A=30°, c=10, a=4$; (*ii*) $A=30°, c=10, a=5$;

(*iii*) $A=30°, c=10, a=5\sqrt{2}$;

(*iv*) $A=30°, c=10, a=10$;

(*v*) $A=60°, c=10, a=10\sqrt{3}$,

(*vi*) $A=120°, c=10, a=5$;

(*vii*) $A=120°, c=10, a=10\sqrt{3}$.

Sol. (*i*) Here the angle is opposite to the smaller side.

From the Sine formula, $\text{Sin C}=\dfrac{c \text{ Sin A}}{a}=\dfrac{10 \text{ Sin } 30°}{4}=\dfrac{5}{4}$.

Thus, Sin C>1 which is impossible, and there is ***no solution.*** [§ 10·9 (*a*) (*i*)-]

(*ii*) The given angle is opposite to the smaller side.

We know, as before, $\text{Sin C} = \dfrac{c \text{ Sin A}}{a} = \dfrac{10 \text{ Sin } 30^\circ}{5} = 1$

$\therefore C = 90^\circ$ or its supplement which is also 90°

Now, $A = 30^\circ$, $C = 90^\circ$. $\therefore B = 180^\circ - (30^\circ + 90^\circ) = 60^\circ$.

and $b = \dfrac{c \text{ Sin B}}{\text{Sin C}} = \dfrac{10 \text{ Sin } 60^\circ}{\text{Sin } 90^\circ} = 5\sqrt{3}$.

The $\triangle$ is rt. angled, only *one solution*; the two solutions are coincident. [§ 10·9 (A) (*ii*)]

(*ii*) The given angle is opposite to the smaller side.

Again, $\text{Sin C} \dfrac{c \text{ Sin A}}{a} = \dfrac{10 \text{ Sin } 30^\circ}{5\sqrt{2}} = \dfrac{1}{\sqrt{2}}$.

Here $C = 45^\circ$ or its supplement 135°. Let them be C_1 and C_2; and since $A + C_1 = 33^\circ + 45^\circ = 75^\circ$.

$A + C_2 = 30^\circ + 135^\circ = 165^\circ$, each is less than 180°; both the values of C are valid and so there are *two solutions*. This is the ambiguous case.

$B_1 = 180^\circ - (A + C_1) = 180^\circ - 75^\circ = 105^\circ$,

$B_2 = 180^\circ - (A + C_2) = 180^\circ - 165^\circ = 15^\circ$,

$$b_1 = \frac{a \text{ Sin } B_1}{\text{Sin A}} = \frac{5\sqrt{2} \text{ Sin } 105^\circ}{\text{Sin } 30^\circ} = \frac{5\sqrt{2}}{\frac{1}{2}} \cdot \frac{\sqrt{3}+1}{2\sqrt{2}}$$

$$= 5(\sqrt{3}+1)$$

$$b_2 = \frac{a \text{ Sin } B_2}{\text{Sin A}} = \frac{5\sqrt{2}. \text{ Sin } 15^\circ}{\text{Sin } 30^\circ} = \frac{5\sqrt{2}.}{\frac{1}{2}} \frac{(\sqrt{3}-1)}{2\sqrt{2}}$$

$$= 5(\sqrt{3}+1).$$

The solutions are

(*i*) $C_1 = 45^\circ$, $B_1 = 105^\circ$, $b_1 = 5(\sqrt{3}+1)$.

(*ii*) $C_2 = 135^\circ$, $B_2 = 15^\circ$, $b_2 = 5(\sqrt{3}-1)$.

[§ 10·9 (A) (*iii*) (*c*)]

(*iv*) Again, $\text{Sin C} = \dfrac{c \text{ Sin A}}{a} = \dfrac{10 \text{ Sin } 30^\circ}{10} = \dfrac{1}{2}$.

$\therefore$ C=30° or 150°, the second value of C is invalid as A+C=30° + 150° must be less than 180°. Hence, there is *only one solution* the triangle is *Isosceles* ; A=C=30°, B=120° and $b=10\sqrt{3}$. [§ 10·9 (*a*) (*iii*) (*b*)]

(*v*) Here the given angle is opposite to the greater side.

$$\text{Again, Sin C} = \frac{c \text{ Sin A}}{a} = \frac{10 \text{ Sin } 60^\circ}{10\sqrt{3}} = \frac{1}{2}$$

$\therefore$ C=30° or its supplement 150°.

The obtuse value of C (150°) is rejected.

Hence, C+A=150°+60°=210° is greater than 180°.

There is, therefore, *only one solution.*

A=60°, C=30°, B=90° and b=20°.

[§ 10·9 (*a*) (*iii*) (*a*)]

(*vi*) The angle is obtuse and is opposite to the smaller side.

$$\text{Sin C} = \frac{c \text{ Sin A}}{a} = \frac{10 \text{ Sin } 120^\circ}{5} = \sqrt{3}.$$

Since Sin C > 1, no value of C is valid, hence the triangle is *impossible.* [§ 10·9 (*b*)]

(*vii*) The angle is obtuse and is opposite to the greater side.

$$\text{Sin C} = \frac{c \text{ Sin A}}{a} = \frac{10 \text{ Sin } 120^\circ}{10\sqrt{3}} = \frac{1}{2}$$

$\therefore$ C=30° or its supplement 150°.

The value 150° is invalid as C+A (=270°) is greater than 180°. C=30° is the only valid solution. Hence there is *only one triangle.*

B=180°−(A+C)=180°−(120°+30°)=30°.

$$\text{C}=30^\circ, \quad b = \frac{10\sqrt{3}}{3}. \qquad [\ \S\ 10.9\ (b)\ ]$$

Ex. 14. If a=5 ft., b=8 ft. and A=35°, find approximately the smaller value of c, having given log 2=·301030,

L Sin 35°=9·758591, L Sin 31° 35′42″=9·719258,

L Sin 66°35′=9·962672, L Sin 66°36′=9·962727.

log 456706=5·659637.

Here,

$$\text{Sin B} = \frac{b \text{ Sin A}}{a} = \frac{8 \text{ Sin } 35^\circ}{5} = \frac{2^4}{10} \text{Sin } 35^\circ \; ;$$

$$\therefore \text{ L Sin B} = 4 \log 2 - \log 10 + \text{L Sin } 35^\circ$$
$$= 1{\cdot}20412 - 1 + 9{\cdot}758591$$
$$= 9{\cdot}962711$$

Now,

L Sin B	$=9{\cdot}962711$	L Sin $66^\circ 36'$	$=9{\cdot}962727$
L Sin $66^\circ 35'$	$=9{\cdot}962672$	L Sin $66^\circ 35'$	$=9{\cdot}962672$
	Diff. $=39$	diff. for $1' =$	55

$$= \frac{39 \times 60''}{55} = 42'' \text{ nearly.}$$

$\therefore \text{B} = 66^\circ 35' 42''$ or $180^\circ - 66^\circ 35' 42''$

i.e. $\text{B}_1 = 66^\circ 35' 42''$, $\text{B}_2 = 113^\circ 24' 18''$.

Hence,

$$\text{C}_1 = 110^\circ - (\text{B}_1 + \text{A}) = 180^\circ - (66^\circ 35' 42'' + 35^\circ)$$
$$= 180^\circ - 101^\circ 35' 42'' = 78^\circ 24' 18'',$$
$$\text{C}_2 = 180^\circ - (\text{B}_2 + \text{A}) = 180^\circ - (113^\circ 24' 18'' + 35^\circ)$$
$$= 180^\circ - 148^\circ 24' 18'' = 31^\circ 35' 42''.$$

$\therefore$ The required side c is opposite to C_2 in the triangle $\text{A B}_2 \text{C}_2$.

Now, from the Sine formula

$$c = \frac{a \text{ Sin C}_2}{\text{Sin A}} = \frac{5 \text{ Sin } 31^\circ 35' 42''}{\text{Sin } 35^\circ},$$

$$\therefore \log c = \log 10 - \log 2 + \text{L Sin } 31^\circ 35' 42'' - \text{L Sin } 35^\circ$$
$$= 1 - {\cdot}30103 + 9{\cdot}719258 - 9{\cdot}758591$$
$$= 10{\cdot}719258 - 10{\cdot}059621$$
$$= {\cdot}659637 = \log 4{\cdot}56706,$$
$$\therefore c = 4{\cdot}56706 \text{ ft. nearly.}$$

EXERCISE XXIV

1. In a triangle ABC, if $a=20$, $c=30$, L Sin A $=9{\cdot}5228787$, find C, log $3={\cdot}4771213$.

2. Find out which of the following data give the ambiguous solution and why ?

   (*i*) $A=30^\circ$, $a=200$ ft., $c=250$ ft.

   (*ii*) $A=30^\circ$, $a=200$ ft., $c=125$ ft.

   (*iii*) $A=30^\circ$, $a=125$ ft., $c=250$ ft.

   Find the smaller value of the third side in the ambiguous case, and third side and other angles in all the cases.

   Given log $2={\cdot}30103$, L Sin $8^\circ41'=9{\cdot}1789001$,

   log $6{\cdot}03893={\cdot}7809601$, L Sin $180^\circ12'40''=9{\cdot}49485$,

   L Sin $38^\circ41'=9{\cdot}7958800$ [*Patna 1942, Alld. 1935.*]

3. If $a=9$, $b=12$, $A=30^\circ$, find c, having given

   log $2={\cdot}30103$, log $3={\cdot}47712$ log $171=2{\cdot}23301$,

   log $368=2{\cdot}56635$, L Sin $11^\circ48'39''=9{\cdot}31108$,

   L Sin $41^\circ48'39''=9{\cdot}82391$, and L Sin $108^\circ11'21''$ $=9.97774$.

4. Find the other angles of a triangle when one angle is $112^\circ4'$, the side opposite to it is 573 ft. long, and another side is 394 ft. long, given log $5{\cdot}73={\cdot}7581564$, log $3{\cdot}94={\cdot}5954962$. L Cos $22^\circ4'=9{\cdot}9669614$.

   L Sin $39^\circ35'=9{\cdot}8042757$, L Sin $39^\circ36'=9{\cdot}8044284$. [*Alld. 1939*]

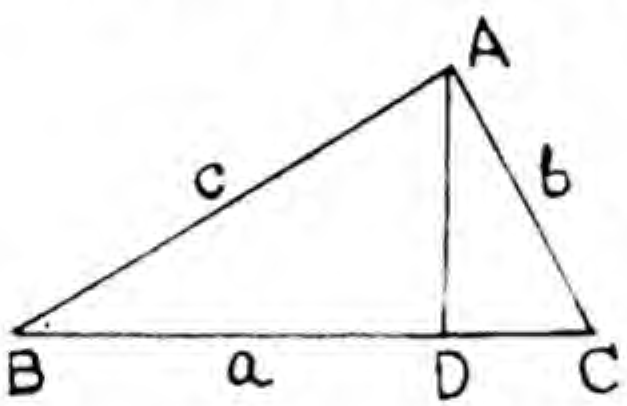
5. If $a=8$, $b=12{\cdot}5$ and $A=33^\circ15'$, show that the triangle has two solutions and find out other angles, given log $2={\cdot}30103$, L Sin $33^\circ15'=3{\cdot}73901$, L Sin $58^\circ50'$ $=9{\cdot}93230$, Diff. for $10'={\cdot}00077$.

CHAPTER XII

(*a*) **Areas of a triangle, regular Polygon and a circle.**

(*b*) **Graphs of Simple Trigonometrical Functions.**

12.1. Area of a triangle.



Let ABC be the given triangle, such that BC $= a$, CA $= b$, and AB $= c$. Draw AD $\perp$ to BC.

Now area of the $\triangle$ ABC $=$

$= \frac{1}{2}$ BC.AD

$= \frac{1}{2}.\ a.$ AD .... (i)

But $\frac{AD}{b} = \text{Sin C}$ $\therefore$ AD $= b$ Sin c

$\therefore$ From (*i*) we get $\triangle$ ABC $= \frac{1}{2}.\ a.\ b$ Sin C

Similarly, we can show that $\triangle = \frac{1}{2}bc$ Sin A.

$= \frac{1}{2}\ ca$ Sin B.

$$\text{Now Sin A} = 2\ \text{Sin}\ \frac{A}{2}\ \text{Cos}\ \frac{A}{2}$$

$$= 2\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{s(s-a)}{bc}}$$

$$= \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$$

$$\therefore \triangle = \tfrac{1}{2}\ bc\ \text{Sin A} = \tfrac{1}{2}\ .\ bc.\ \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{s(s-a)(s-b)(s-c)}$$

$$\text{Again, } b = \frac{a\ \text{Sin B}}{\text{Sin A}} \text{ and } c = \frac{a\ \text{Sin C}}{\text{Sin A}}$$

$$\left[\because \frac{a}{\text{Sin A}} = \frac{b}{\text{Sin B}} = \frac{c}{\text{Sin C}}\right]$$

$$\therefore \triangle = \tfrac{1}{2}\, bc \text{ Sin A} = \tfrac{1}{2} . \frac{a \text{ Sin B}}{\text{Sin A}} . \frac{a \text{ Sin C}}{\text{Sin A}} . \text{Sin A}$$

$$= \tfrac{1}{2} \frac{a^2 \text{ Sin B Sin C}}{\text{Sin A}}$$

$$\text{Similarly, } \triangle = \tfrac{1}{2} \frac{b^2 \text{ Sin C Sin A}}{\text{Sin B}}$$

$$\text{and } \triangle = \tfrac{1}{2} \frac{c^2 \text{ Sin A Sin B}}{\text{Sin C}}$$

Hence the area of a $\triangle$ ABC can be put in any of the following three ways :—

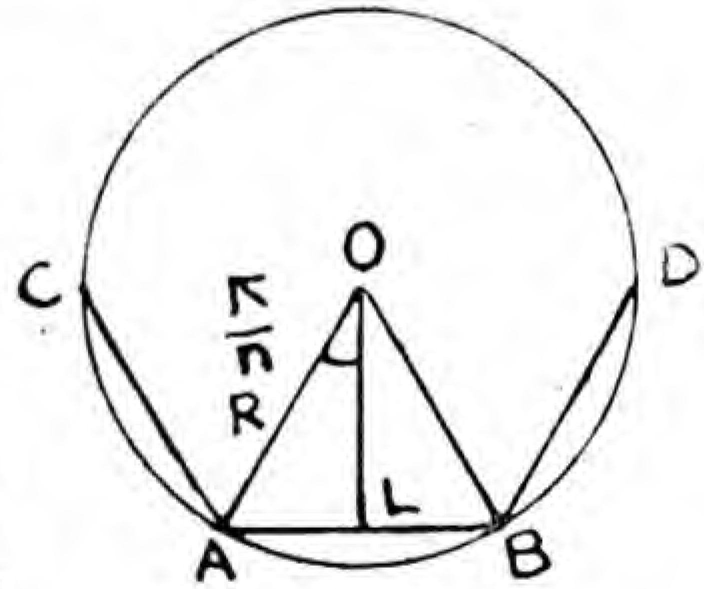
1. $\triangle = \tfrac{1}{2} ab \text{ Sin C} = \tfrac{1}{2}\, bc \text{ Sin A} = \tfrac{1}{2}\, ca \text{ Sin B}$
[one half product of *any* two sides and the included angle]

2. $\triangle = \sqrt{s(s-a)(s-b)(s-c)}$ where $a+b+c=2s$

3. $\triangle = \tfrac{1}{2} \dfrac{a^2 \text{ Sin B Sin C}}{\text{Sin A}} = \tfrac{1}{2} . \dfrac{b^2 \text{ Sin C Sin A}}{\text{Sin B}} = \tfrac{1}{2} . \dfrac{c^2 \text{ Sin A Sin B}}{\text{Sin C}}$

12.2 Def—Regular polygon :—A polygon whose sides and angles are equal is called a **Regular Polygon.**

12·2.1. Radius of the circumscribed circle of a Regular plygon of n sides



Let CABD......be the regular Polygon of n sides. Draw OA and OB as bisectors of angles A and B. Let them meet at O. From O draw OL$\perp$ to AB. Then if AB$=a$, AL $= \dfrac{a}{2}$. Now O is the centre of the circumscribed circle, such that OA=OB=R is the circum radii. As the number of sides is n, $\angle$AOB$=$ $\dfrac{2\pi}{n}$.

$$\therefore \angle \text{AOL} = \tfrac{1}{2} . \frac{2\pi}{n} = \frac{\pi}{n}$$

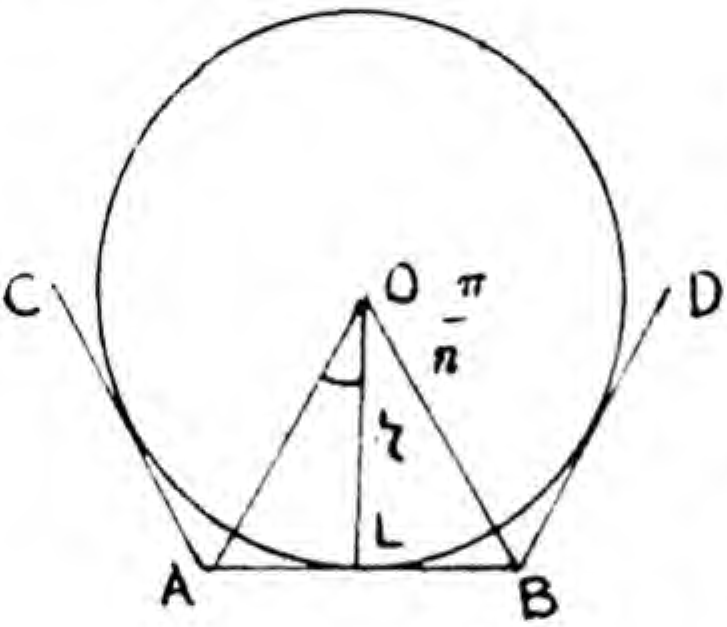
Now from the rt. $\angle d$ $\triangle$AOL, we have

$$\frac{AL}{AO} = \text{Sin AOL}$$

or
$$\frac{a}{2R} = \text{Sin } \frac{\pi}{n}$$

$$\therefore \quad R = \frac{a}{2} \text{Cosec } \frac{\pi}{n}$$

12.2.2 Radius of the inscribed circle of a regular Polygon.



As before, draw **OA** and **OB** bisectors of angles **A** and **B** so as to meet at O. Then O is the centre of the circle. Draw OL $\perp$ to AB. Then OL$=r$ is the radius of the inscribed circle, and

$$AL = \frac{a}{2} \text{ if } AB = a$$

Now from the rt. $\angle$d. $\triangle$ AOL, we have

$$\frac{AL}{OL} = \tan \text{AOL}$$

$$\therefore \quad \frac{a}{2r} = \tan \frac{\pi}{n} \text{ which gives}$$

$$r = \frac{a}{2} \cot \frac{\pi}{n}$$

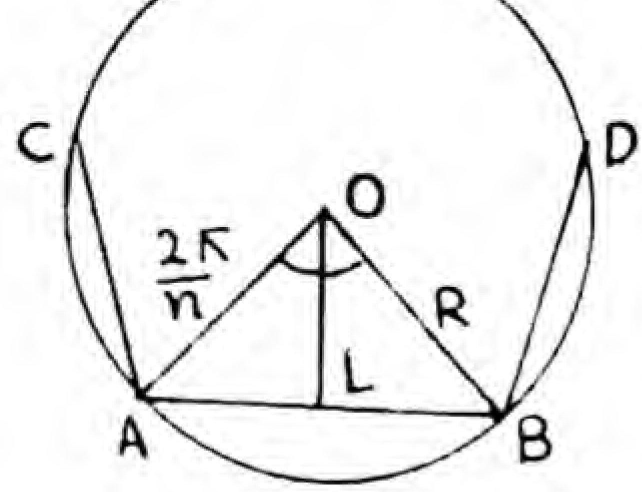
12.3.1 Area of a Regular Polygon in terms of R (circum-radius)

Area of the whole polygon of n sides $= n$ times the area of the $\triangle$AOB

$= n \times \frac{1}{2} \times OA \times OB \times \text{Sin AOB}$ (article 12·1)

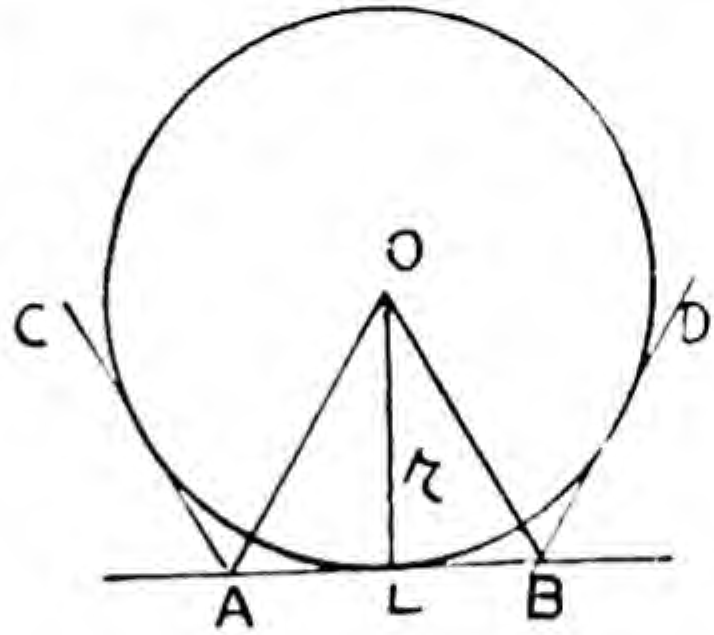
$n \times \frac{1}{2} \times R \times R \times \text{Sin } \frac{2\pi}{n}$

$= \frac{n}{2} R^2 \; Sin \; \frac{2\pi}{n}$



12.3.2. Area of a Regular polygon in terms of its side and r.

(*i*)



The whole area $= n$ time the area of the $\triangle AOB$

$= n \times \frac{1}{2} \times OL \times AB$

($\because \frac{1}{2} \times \text{base} \times \text{altitude}$)

$= \frac{1}{2}.\ n.\ r.\ a.$

$= \frac{1}{2}.\ n.\ a.\ \frac{a}{2} \cot \frac{\pi}{n}$

$\left(\because r = \frac{a}{2} \text{Cot } \frac{\pi}{n}\right)$

$$= \frac{na^2}{4} \text{ Cot } \frac{\pi}{n}$$

(*ii*) Again, area $= \frac{n}{2} \times OL \times AB$

$$= \frac{n}{2} \times r \times 2r \tan \frac{\pi}{n}$$

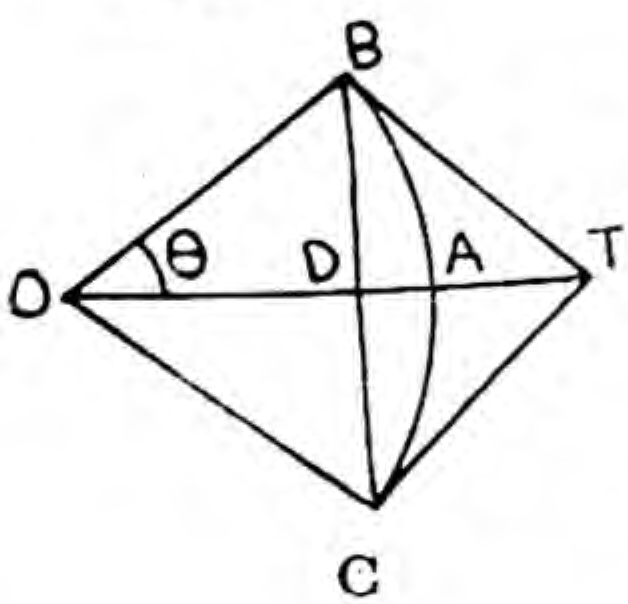
$$= nr^2 \tan \frac{\pi}{n}$$

12.4 An Important Limit

To prove that $\underset{\theta \to 0}{\text{Lt}} \frac{\text{Sin } \theta}{\theta} = 1$ where

θ is measured in radians

Proof.



Let $\angle AOB = \theta$ radians, when AB is the arc of the circle whose radius is OB, and the centre is O.

Draw BD $\perp$ to OA and produce it to meet the arc of the circle again in **C.**

At B and C draw tangents to the circle to meet OA produced in T.

Now $\quad BDC < \text{arc } BAC \quad < \text{?}\ (BT+TC)$
or $\quad 2BD \ < 2 \text{ arc } \ BA \ < 2\ BT$
or $\quad BD \ < \text{arc } \ BA \ < BT$
Dividing by OB, we get

$$\frac{BD}{OB} < \frac{\text{arc } BA}{OB} < \frac{BT}{OB}$$

But $\dfrac{BD}{OB} = \text{Sin } \theta$

$\dfrac{\text{arc } BA}{BA} = \theta \qquad\qquad \left(\because \dfrac{l}{r} = \theta\right)$

and $\dfrac{BT}{OB} = \tan \ \theta$

$\therefore$ we get $\text{Sin } \theta < \theta < \tan \theta$

Dividing by $\text{Sin } \theta$, we have

$$1 < \frac{\theta}{\text{Sin } \theta} < \frac{1}{\text{Cos } \theta}$$

which shows that $\dfrac{\theta}{\text{Sin } \theta}$ lies between 1 and $\dfrac{1}{\cos \theta}$

$$\text{But } \underset{\theta \to 0}{\text{Lt}} \frac{1}{\text{Cos } \theta} = \frac{1}{1} = 1$$

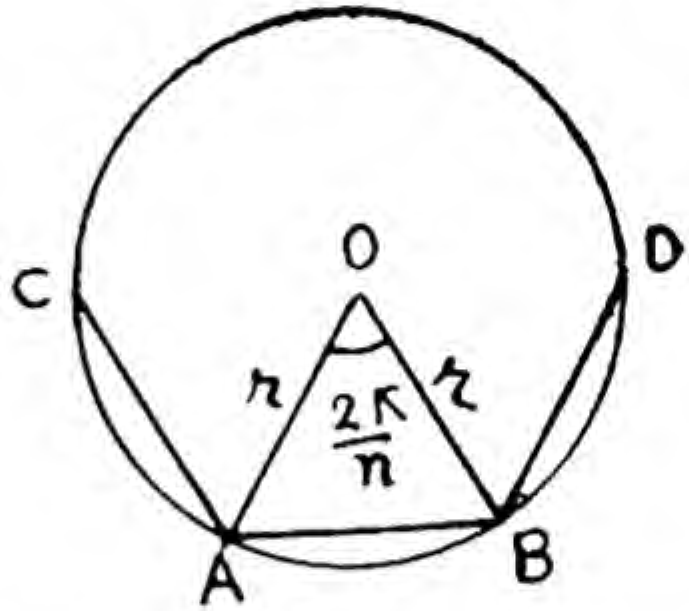
$$\therefore \underset{\theta \to 0}{\text{Lt}} \frac{\theta}{\text{Sin } \theta} \text{ lies between 1 and 1}$$

$$\text{or} \quad \underset{\theta \to 0}{\text{Lt}} \frac{\theta}{\text{Sin } \theta} = 1$$

$$\text{or} \quad \underset{\theta \to 0}{\text{Lt}} \frac{\text{Sin } \theta}{\theta} = 1$$

12.4.1. Area of a circle of radius r and its circumference.

(*i*) Let AB be the side of a regular polygon of n sides inscribed in a circle of radius r and centre O.



Then area of the polygon $= n$ times the area of the $\triangle$AOB

$= n.\frac{1}{2}.\, r.\, r.\, \text{Sin AOB}$

$= \frac{nr^2}{2} \text{Sin } \frac{2\pi}{n}$

If the number of the sides of the polygon is increased indefinitely, this area becomes the area of the circle.

$\therefore$ The area of the circle $= \underset{n \to \infty}{\text{Lt}} \frac{\pi r^2}{2} \text{Sin } \frac{2\pi}{n}$

$$= \underset{n \to \infty}{\text{Lt}} \frac{nr^2}{2} \cdot \frac{2\pi}{n} \cdot \frac{\text{Sin } \frac{2\pi}{n}}{\frac{2\pi}{n}}$$

(Please note this step)

$$= \text{Lt}\, \pi r^2 . \frac{\text{Sin } \frac{2\pi}{n}}{\frac{2\pi}{n}}$$

$\left[\because \text{If } n \to \infty, \frac{2\pi}{n} \to 0\right.$

as $\frac{2\pi}{n} \to 0$

$= \pi r^2 \qquad \left(\because \underset{\theta \to 0}{\text{Lt}} \frac{\text{Sin } \theta}{\theta} = 1\right)$

(*ii*) Perimeter of the polygon $= n.\ AB$

$$= n.\ 2r\ \text{Sin}\frac{\pi}{n}$$

$$= 2nr\ \text{Sin}\ \frac{\pi}{n}$$

This perimeter will become the circumference of the circle if the number of sides increases indefinitely.

$$\therefore \text{ the circumference} = \underset{n\to\infty}{\text{Lt}}\ 2nr\ \text{Sin}\ \frac{\pi}{n}$$

$$= \text{Lt}\ 2\ nr.\ \frac{\pi}{n}\ \frac{\text{Sin}\frac{\pi}{n}}{\frac{\pi}{n}}$$

Please note this step)

$$= 2\pi r$$

$$\left(\because \underset{\frac{\pi}{n}\to 0}{\text{Lt}}\ \frac{\text{Sin}\frac{\pi}{n}}{\frac{\pi}{n}} = 1\right)$$

Solved examples

Ex. 1. If R, r be the radii of the circumcircle and the circle inscribed in a regular polygon of n sides, each side being of length a, prove that

$$R + r = \tfrac{1}{2}\ a\ \text{Cot}\left(\frac{\pi}{2n}\right)$$

(K.U. *Pre*—1962)

Sol. We have $R = \dfrac{a}{2\ \text{Sin}\ \frac{\pi}{n}}$ and $r = \dfrac{a}{2\ \tan\frac{\pi}{n}}$

$$\therefore\ R + r = \tfrac{1}{2}.a.\left(\frac{1}{\text{Sin}\ \frac{\pi}{n}} + \frac{1}{\tan\ \frac{\pi}{n}}\right)$$

$$=\frac{a}{2}\left(\frac{1+\text{Cos}\ \frac{\pi}{n}}{\text{Sin}\ \frac{\pi}{n}}\right)=\frac{a}{2}\cdot\frac{2\ \text{Cos}^2\frac{\pi}{2n}}{2\ \text{Sin}\ \frac{\pi}{2n}\ \text{Cos}\ \frac{\pi}{2n}}$$

$$=\frac{A}{2}\ \text{Cot}\left(\frac{\pi}{2n}\right)$$

Ex. 2. If an equilateral triangle and a regular hexagon have the same perimeter, prove that their areas are in the ratio of 2 : 3

Sol. Let the perimeter of each $=6a$

$\therefore$ each side of the $\triangle=2a$

and each side of the regular

hexagon $=a$

Now area of the $\triangle=\frac{1}{2}\ bc\ \text{Sin A}$

$$=\tfrac{1}{2}\ 2a.\ 2a.\ \text{Sin}\ 60°$$

$$=2a^2\ \frac{\sqrt{3}}{2}=a^2\sqrt{3}$$

Also area of the hexagon $=\dfrac{6\ (a)^2}{4}\ \text{Cot}\ 30°$

$$\left[\because\ \text{area}=\frac{na^2}{4}\text{Cot}\ \frac{\pi}{n}\right]$$

$$=\frac{3}{2}\cdot a^2.\sqrt{3}=\frac{3\sqrt{3}}{2}\ a^2$$

$$\therefore\quad \text{ratio}=\frac{\sqrt{3}a^2}{\frac{3\sqrt{3}a^2}{2}}=\frac{2}{3}$$

CHAPTER VII (*Continued*)

Variations of Trigonometrical Ratios and their Graphs

12.5. *To trace the variations of Sin θ as θ increases continuously from $0°$ to $360°$, and to exhibit them graphically.*

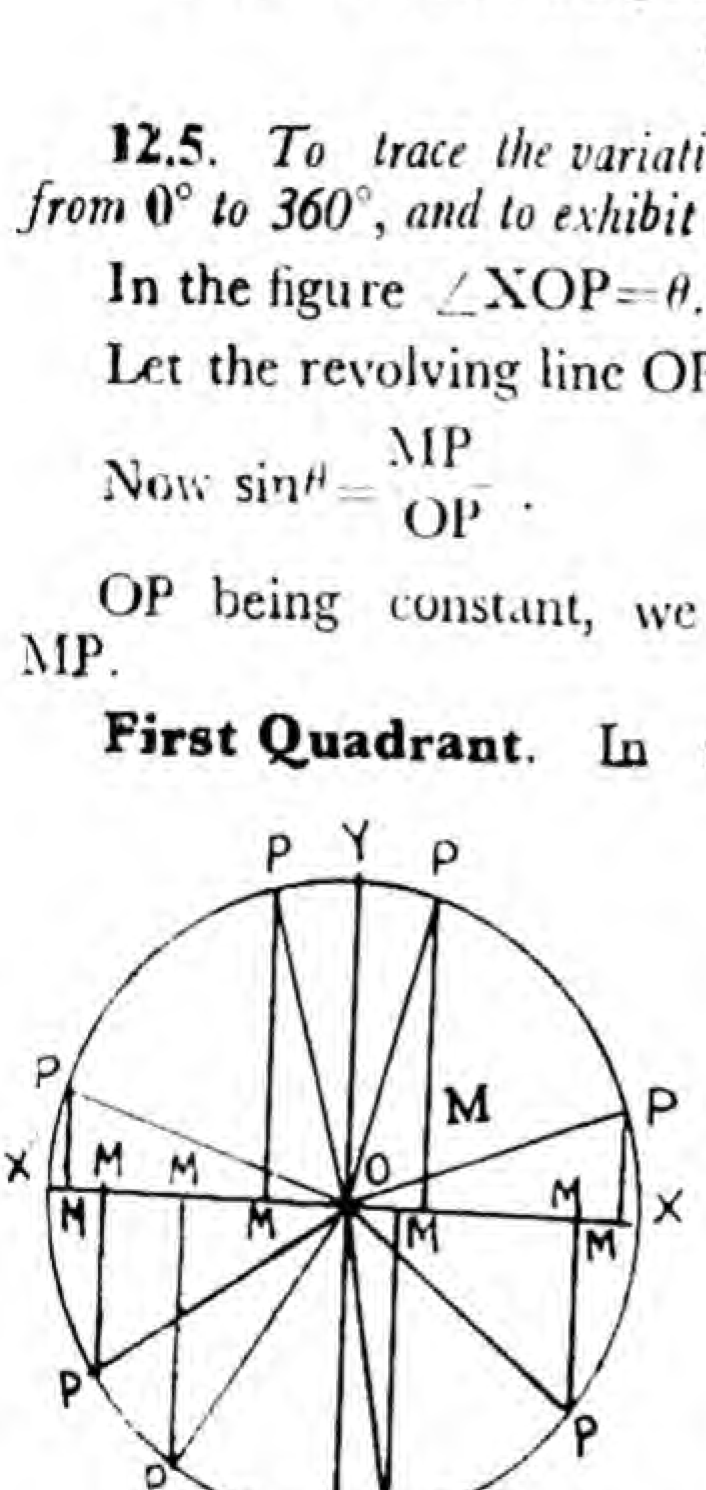
In the figure $\angle XOP=\theta$.

Let the revolving line OP be of constant length, say 1.

Now $\sin\theta=\dfrac{MP}{OP}$.

OP being constant, we have to observe the variations of MP.

First Quadrant. In the first quadrant when $\theta=0°$ M and P coincide and therefore MP is zero, so that Sin $0°=0$. As θ increases, MP and therefore Sin θ increases, till when $\theta=90°$ MP=OP and hence Sin $90°=1$ Thus in the first quadrant as θ varies from $0°$ to $90°$, Sin θ is positive and varies from 0 to 1, *i.e.*, increases from 0 to 1.



Second Quadrant. As θ increases, MP is positive and decreases so that Sin θ is positive and decreases: and when $\theta=180°$, MP vanishes and therefore Sin $180°=0$.

Thus in the second quadrant Sin θ varies from 1 to 0 *i.e.*, decreases from 1 to 0 and is positive because MP is positive.

Third Quadrant. As θ increases, MP is negative and increases in magnitude so that Sin θ is negative and increases in magnitude.

When $\theta=270°$, MP $=$ OP in magnitude and $\therefore$ Sin $270°=-1$.

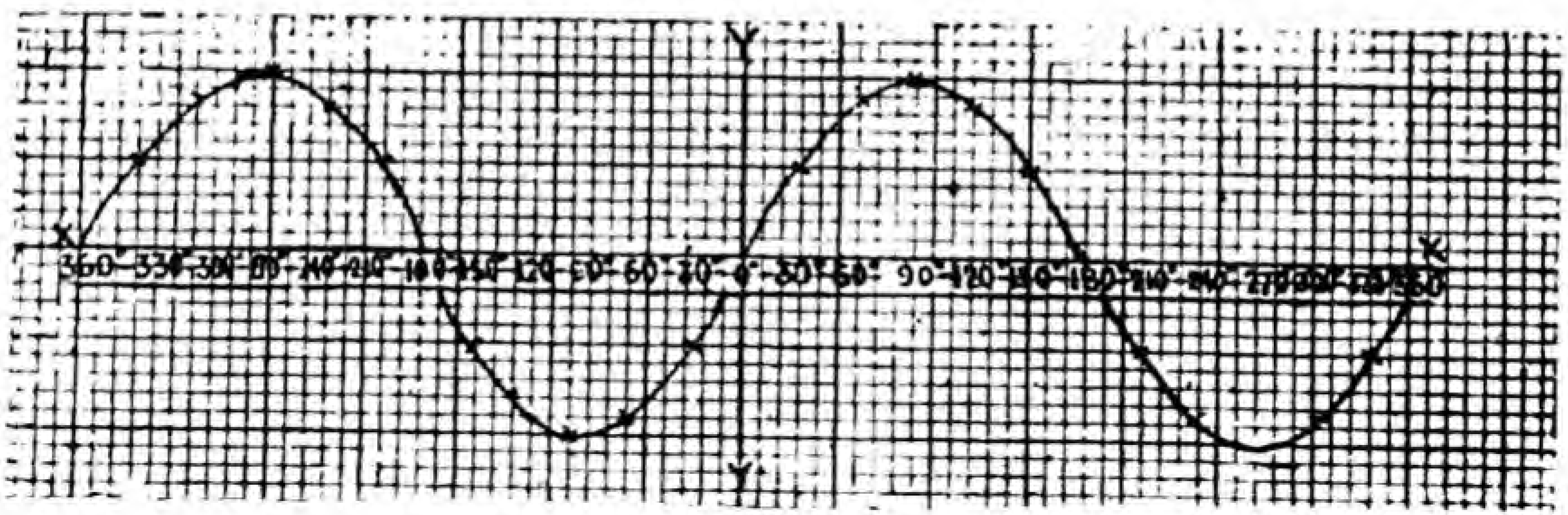
Thus in the third quadrant Sin θ varies from 0 to -1 and is negative because MP is negative.

Fourth Quadrant. As θ increases, MP is negative and decreases in magnitude, so that Sin θ is negative and decreases in magnitude. When $\theta=360°$, MP is zero, so that sin $360°=0$.

Thus in the fourth quadrant Sin θ varies from -1 to 0 and is negative, because MP is negative.

TABLE FOR THE SINE GRAPH

$x =$	-360°	-330°	-300°	-270°	-240°	-210°	-180°	-150°	-120°	-90°	-60°	-30°	0°
$\sin x =$	0	·5	·87	1	·87	5	0	−·5	−·87	−1	−·87	−·5	0
$x =$	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$\sin x =$	·0	·5	·87	1	·87	·5	·0	−·5	−·87	−1	−·87	−·5	0



The Sin Graph

12 6. *To trace the variations of Cos θ as θ varies continuously from 0° to 360° and to exhibit them graphically.*

Referring to the figure of Article 12.5, $\text{Cos}\,\theta = \dfrac{OM}{OP}$.

So the variations in Cos θ depend upon the variations in the values of OM.

First Quadrant. In the first quadrant when $\theta = 0°$, M and P coincide and therefore OM=OP and hence Cos 0°=1. As θ increases, OM and therefore Cos θ decreases, till when $\theta = 90°$, OM is zero, and hence Cos 90°=0.

Thus in the first quadrant Cos θ varies from 1 to 0, *i. e.*, decreases and is positive, because OM is positive.

Second Quadrant. As θ increases, OM is negative and increases in magnitude, consequently Cos θ is negative and increases in magnitude, till when $\theta = 180°$, OM=OP in magnitude and hence Cos 180°=−1.

Thus in the second quadrant Cos θ varies from 0 to −1 and is negative because OM is negative.

Third Quadrant. As θ increases, OM is still negative and decreases in magnitude ; so that Cos θ is negative and decreases in magnitude, till when $\theta = 270°$, OM is zero and therefore Cos 270°=0.

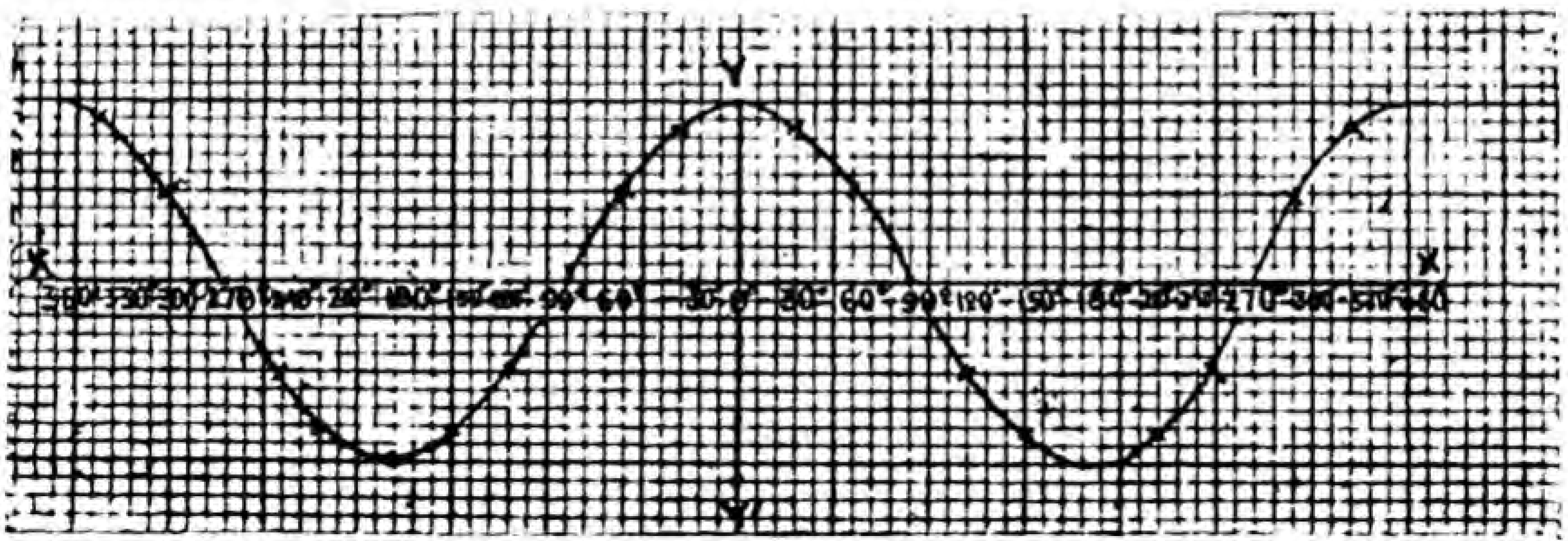
Thus in the third quadrant Cos θ varies from −1 to 0 and is negative, because OM is negative.

Fourth Quadrant. As θ increases, OM is positive and increases so that Cos θ is positive and increases, till when $\theta = 360°$, OM=OP, and therefore Cos 360°=1,

Thus in the fourth quadrant Cos θ varies from 0 to 1 and is positive, because OM is positive.

TABLE FOR THE COSINE GRAPH

$x =$	$-360°$	$-330°$	$-300°$	$-270°$	$-240°$	$-210°$	$-180°$	$-150°$	$-120°$	$-90°$	$-60°$	$-30°$	$0°$
$\cos x =$	1	·87	·5	0	−·5	−·87	−1	−·87	−·5	0	·5	·87	1
$x =$	$0°$	$30°$	$60°$	$90°$	$120°$	$150°$	$180°$	$210°$	$240°$	$270°$	$300°$	$300°$	$3·60°$
$\cos x =$	1	·87	·5	0	−·5	−·87	−1	−·87	−·5	0	·5	·87	1



Cosine Graph

12.7 *To trace the variations of tan θ as θ varies continuously from 0° to 360° and to exhibit them graphically.*

Referring to the figure of article 12·5 $\tan\theta = \dfrac{MP}{OM}$.

So the variations in tan θ depend upon the variations in both MP and OM.

First Quadrant. In the first quadrant when θ=0°, M and P coincide so that MP is zero and OM=OP and therefore tan 0°=0.

As θ increases, MP increases, and OM decreases and therefore on both these accounts tan θ increases. When OP has turned through an angle which is slightly less than a right angle so that P is very near to Y, OM is very small and MP is very nearly equal to OP or 1 and consequently tan θ is very large ; therefore by taking an angle sufficiently near to 90° we can make the tangent as large as we please. This fact is, for the sake of brevity expressed thus : the tangent of 90° is infinite.

In the first quadrant, therefore, tan θ increases from 0 to ∞ (infinity), and is positive, because MP and OM are both positive.

Second Quadrant. As θ increases slightly, OM becomes negative while remaining small, and MP is positive and very nearly equal to OP or 1, so that the corresponding tangent is very large and negative. As θ increases in magnitude, OM increases in magnitude while MP decreases, so that tan θ decreases in magnitude, till when θ=180°. MP is zero, and OM=OP=1 and therefore tan 180°=0.

In the second quadrant, therefore, tan θ varies from −∞ to 0 and is negative, because OM is negative and MP is positive.

Third Quadrant. As θ increases, OM and MP both become negative and OM decreases in magnitudes while MP in-

creases in magnitude, so that $\tan\theta$ is positive and increases, till when $\theta\to 270°$; $OM\to 0$ and $MP\to OP=1$ and $\therefore \tan 270°$ is infinite.

In the third quadrant, therefore, $\tan\theta$ varies from O to ∞ and is positive, because OM and MP are both negative.

Fourth quadrant. As θ increses slightly, OM is small but becomes positive, while MP remains negative, and very nearly equal to OP or 1 so that the corresponding tangent is very large and negative. As θ increases, OM increases and MP decreases in magnitude, so that $\tan\theta$ decreases in magnitude, till when $\theta=360°$, MP is zero and $OM=OP=1$ and therefore $\tan 360°=0$.

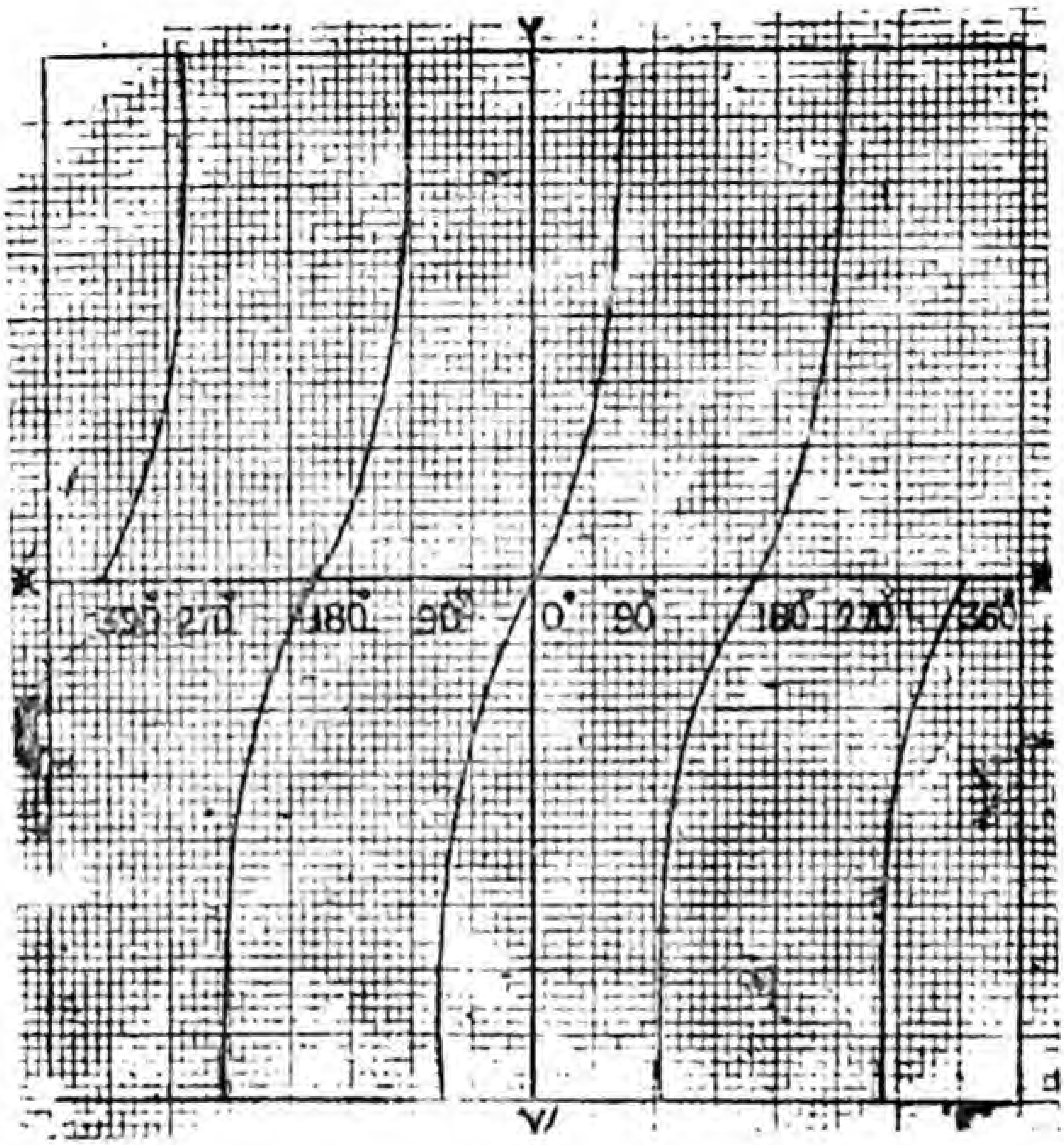
In the fourth quadrant, therefore, $\tan\theta$ varies from $-\infty$ to 0 and is negative, because OM is positive and MP is negative.

Note 1. It follows that $\tan\theta$ is capable of assuming any real value whatever.

Note 2. It also follows that there are two angles lying between 0° and 360°, which have a given tangent; if the given tangent is positive, one of the angles lies between 0° and 90° and the other between 180° and 270°, but if the given tangent is negative, then one of the angles lies bteween 90° and 180° and the other between 270° and 360°.

TABLE FOR THE TANGENT GRAPH

$x=$	$-360°$	$-330°$	$-300°$	$-270°-0°$	$-270°+0°$	$-240°$	$-210°$	$-180°$	$-150°$	$-120°$	$-90°-0°$	$-90°+0°$	$-60°$	$-30°$	$0°$
$\tan x=$	0	·58	1·7	$+\infty$	$-\infty$	−1·7	−·58	0	·58	1·7	$+\infty$	$-\infty$	−1·7	−·58	0
$x=$	$0°$	$30°$	$60°$	$90°-0°$	$90°+0°$	$120°$	$150°$	$180°$	$210°$	$240°$	$270°-0°$	$270°+0°$	$300°$	$330°$	$360°$
$\tan x=$	0	·58	1·7	$+\infty$	$-\infty$	−1·7	−·58	0	·58	1·7	$+\infty$	$-\infty$	−1·7	−·58	0



The Tangent Graph

As θ increases, OM decreases and MP increases; so that Cot θ decreases, till when $\theta=90°$, OM is zero and MP=OP=1 and consequently Cot $90°=0$.

Thus in the first quadrant Cot θ varies from ∞ to 0 and is positive, because OM and MP are both positive.

Second quadrant. As θ increases, OM becomes negative and increases in magnitude, while MP is positive and decreases, so that Cot θ is negative and increases in magnitude, till when θ is very near to 180°, MP is very small and OM is very nearly equal to OP or 1 and, therefore, Cot 180° is negative and infinite.

Thus in the second quadrant Cot θ varies from 0 to $-\infty$ and is negative because OM is negative and MP is positive.

Third quadrant. As θ is slightly greater than 180°, OM and MP both become negative and MP is small, and OM is very nearly equal to OP or 1, so that Cot θ is positive and infinite. As θ increases, MP increases in magnitude while OM decreases in magnitude so that Cot θ is positive and decreases in magnitude, till when $\theta=270°$, OM is zero and MP=OP or 1 and therefore Cot $270°=0$.

Thus in the third quadrant Cot θ varis from $+\infty$ to 0 and is positive, because OM and MP are both negative.

Fourth Quadrant. As θ increases, OM becomes positive and increases while MP is negative and decreases in magnitude, so that θ is negative, and increases in magnitude, till when θ is very near to 360°, MP is small and OM is very nearly equal to OP or 1 and therefore Cot 360° is negative and infinite.

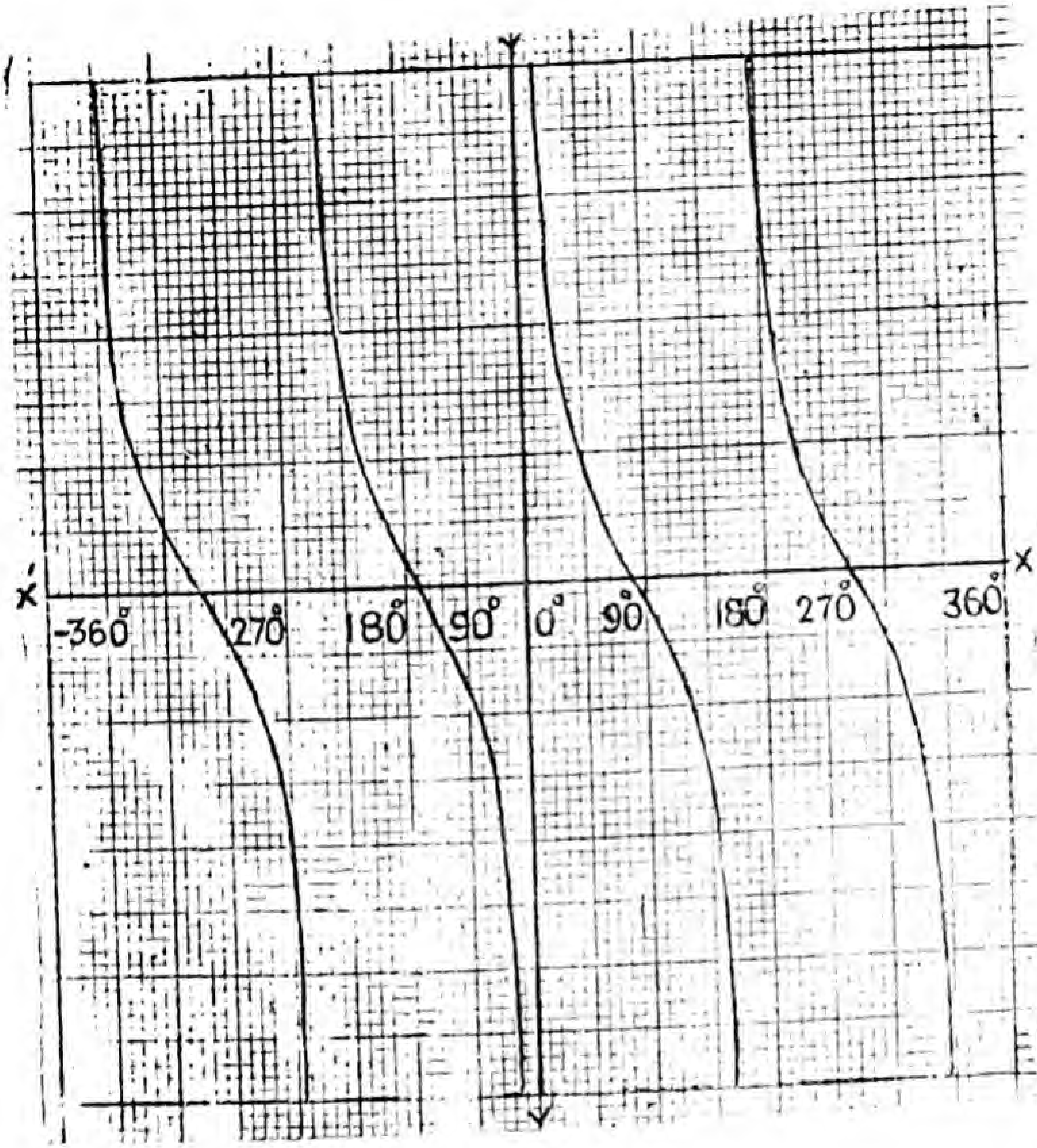
Thus in the fourth quadrant Cot θ varies from 0 to $-\infty$ and is negative, because OM and MP have opposite sign.

Note 1. It follows that Cot θ is capable of assuming any real value whatever.

Note 2. It also follows that there are two angles lying between 0° and 360°, which have a given Cotangent; if the given Cotangent is positive, one of the angles lies between 0° and 90°, and the other between 180° and 270°; but if the given Cotangent is negative then one of the angles lies between 90° and 180° and the other between 270° and 360°.

TABLE FOR THE COTANGENT GRAPH

$x =$	$-360°+0°$	$-330°$	$-300°$	$-270°$	$-240°$	$-210°$	$-180°-0$	$-180°+0$	$-150°$	$-120°$	$-90°$	$-60°$	$-30°$	$\cdot 0°$
$\cot x =$	$+\infty$	1·7	·58	0	−·58	−·17	$-\infty$	$+\infty$	1·7	58	0	−·58	−1·7	$-\infty$
$x =$	$0°$	$30°$	$60°$	$90°$	$120°$	$150°$	$180°$ $-0°$	$180°$ $+0°$	$210°$	$240°$	$270°$	$300°$	$330°$	$360°$ $-0°$
$\cot x =$	$+\infty$	1·7	−·58	0	·58	−1·7	$-\infty$	$+\infty$	1·7	·58	0	−·58	−1·7	$-\infty$



The Co-tangent Graph

12·9 *To trace the variations of Secant θ as θ varies continuously from 0° to 360°, and to exhibit them graphically.*

Referring to the figure of Article 12·5 $\sec\theta = \dfrac{OP}{OM}$.

OP being constant; we have to observe the variations of OM.

First Quadrant. When θ is zero, M and P coincide, so that OM = OP and consequently sec 0° = 1. As θ increases, OM decreases so that Sec θ increases; when θ is very near to 90°, OM is very near to 0 and therefore, Sec 90° is infinite.

Thus in the first quadrant Sec θ varies from 1 to ∞ and is positive because OM is positive.

Second Quadrant. As θ increases slightly, OM becomes negative and remains small, so that Sec θ is negative and infinite. As θ increases, OM increases in magnitude so that Sec θ is negative and decreases in magnitude till when $\theta = 180°$. OM equals OP in magnitude and therefore Sec 180° = −1.

Thus in the second quadrant Sec θ varies from −∞ to −1, is negative, because OM is negative.

Third Quadrant. As θ increases, OM remains negative and decreases in magnitude; so that Sec θ is negative and increases in magnitude; when θ comes nearer and nearer to 270° OM becomes smaller and smaller therefore Sec θ becomes larger and larger; hence Sec 270° is infinite and negative.

Thus in the third quadrant Sec θ varies from −1 to −∞ and is negative, because OM is negative

Fourth Quadrant. As θ increases slightly, OM becomes positive and remains small and therefore sec θ is positive and infinite. As θ increases, OM increases and therefore sec θ decreases till when $\theta = 360°$, OM = OP and therefore Sec 360° = 1.

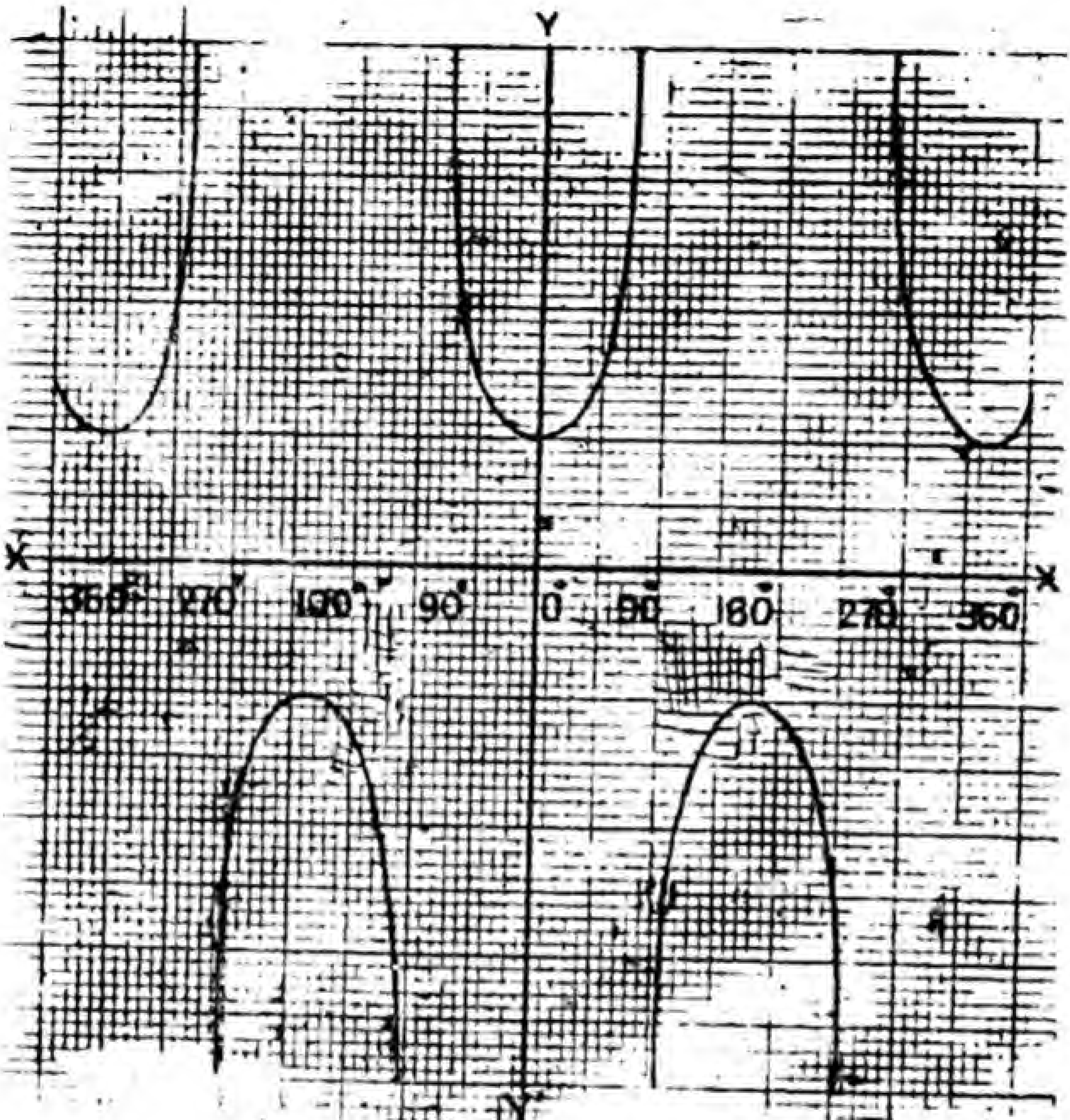
Thus in the fourth quadrant Sec θ varies from ∞ to 1 and is positive because OM is positive.

Note 1. It follows that Sec θ never lies between 1 and −1 and that it is capable of assuming any real value not lying between 1 and −1.

Note 2. It also follows that there are two angles lying between 0° and 360°, which have a given Secant, if the given Secant is positive, one of the angles lies between 0° and 90° and the other between 270° and 360° but if the given Secant is negative, then the angles lie between 90° and 270°.

TABLE FOR THE SECENT GRAPH

$x =$	-360°	-330°	-300°	$-270^\circ-0^\circ$	$-270^\circ+0^\circ$	-240°	-210°	-180°	-150°	-120°	$-90^\circ-0^\circ$	$-90^\circ+0^\circ$	-60°	-30°	0°
$\sec x =$	1	1·2	2	$+\infty$	$-\infty$	-2	$-1{\cdot}2$	-1	$-1{\cdot}2$	-2	$-\infty$	$+\infty$	2	1·2	1
$x =$	0°	30°	60°	$90^\circ-0^\circ$	$90^\circ+0^\circ$	120°	150°	180°	210°	240°	$270^\circ-0^\circ$	$270^\circ+0^\circ$	300°	330°	360°
$\sec x =$	1	1·2	2	$+\infty$	$-\infty$	-2	$-1{\cdot}2$	-1	$-1{\cdot}2$	-2	$-\infty$	$+\infty$	2	1·2	1



The Secant Graph

3.9.1. *To trace the variations of Cosec θ as θ varies continuously from 0 to 360° and to exhibit them graphically.*

Referring to the figure of Art. 12.5

$$\text{Cosec } \theta=\frac{OP}{MP}.$$

OP being constant, we have to observe the variations of MP.

First Quadrant. When θ is very small, MP is positive and very small and as $\theta \to 0$, $MP \to 0$ and $\therefore$ Cosec $\theta \to \infty$, so that Cosec θ is infinite to start with. As θ increases, MP increases and therefore Cosec θ decreases, till when $\theta=90°$. MP equals OP and therefore Cosec $90°=1$.

Thus in the first quadrant Cosec θ varies from ∞ to 1 and is positive because MP is positive.

Second Quadrant. As θ increases, MP is positive and decreases, so that the Cosec θ increases; when θ approaches nearer and nearer to 180°, MP approaches zero, so that Cosec 180° is infinite.

Thus in the second quadrant Cosec θ varies from 1 to ∞ and is positive because MP is positive.

Third Quadrant. As θ increases slightly, MP is small but becomes negative, so that Cosec θ is negative and infinite.

As θ increases, MP increases in magnitude so that Cosec θ decreases in magnitude till when $\theta=270°$, MP equals OP in magnitude and therefore Cosec $270°=-1$.

Thus in the third quadrant Cosec θ varies from ∞ to −1 and is negative, because MP is negative.

Fourth Quadrant. As θ increases, MP remains negative and decreases in magnitude; so that Cosec θ is negative and increases in magnitude. When θ approaches nearer and nearer to 360°, MP approaches zero and therefore Cosec θ becomes larger and larger; hence Cosec 360° is negative and infinite.

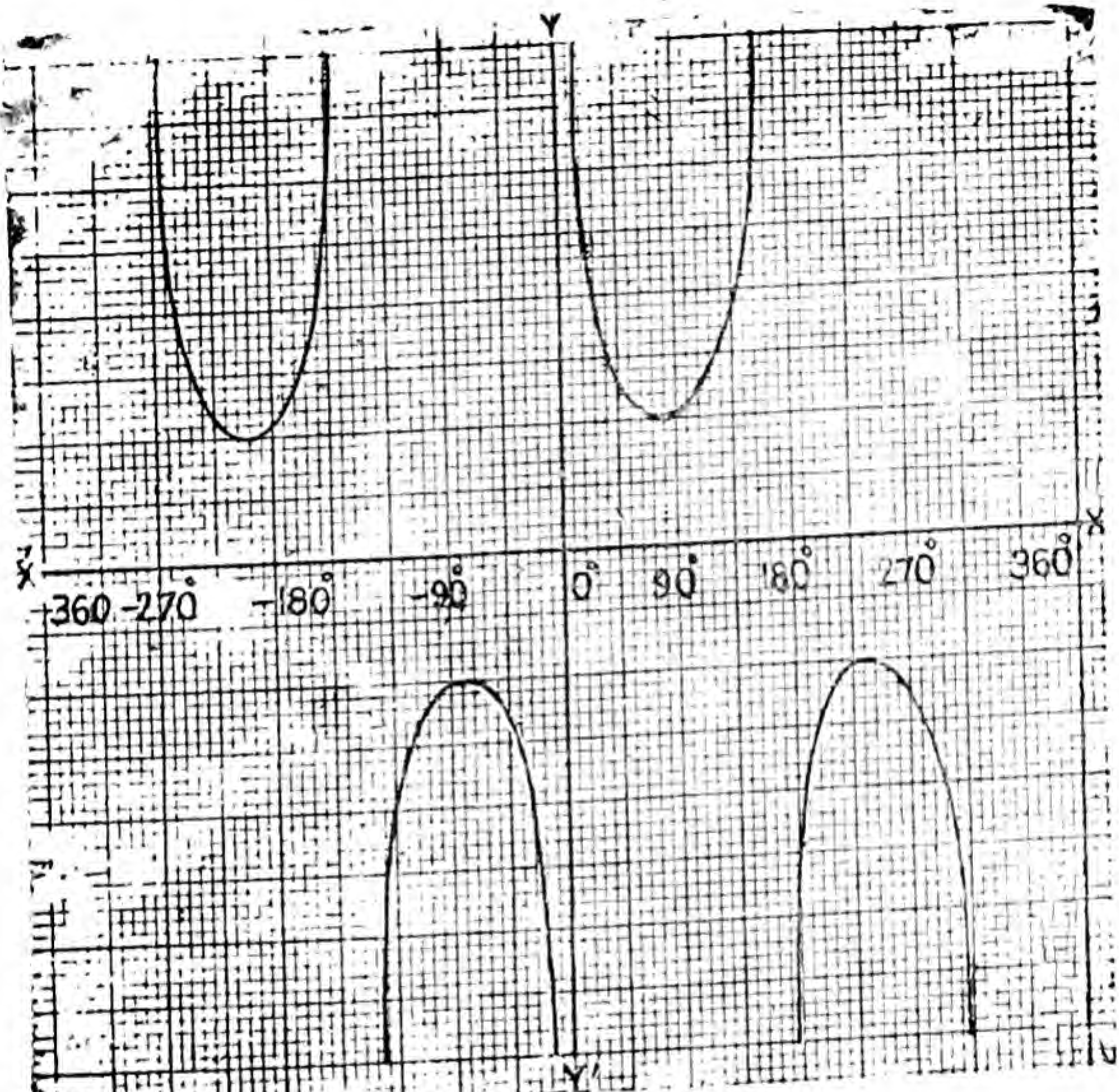
Thus in the fourth quadrant Cosec θ varies from −1 to −∞ and is negative, because MP is negative.

Note 1.—It follows that Cosec θ never lies between 1 and -1 and that it is capable of assuming any real value not lying between 1 and -1.

Note 2.—It also follows that there are two angles lying between 0° and 360° which have a given Cosecant; if the given Cosecant is positive, the angles lie between 0° and 180°; but if the given Cosecant is negative, the angles lie between 180° and 360°.

TABLE FOR THE COSECANT GRAPH

$x =$	$-360°+0$	$-330°$	$-300°$	$-270°$	$-240°$	$-210°$	$-180°-0$	$-180°+0°$	$-150°$	$-120°$	$-90°$	$-60°$	$-30°$	$-0°$
COSEC =	$+\infty$	2	1·2	1	1·2	2	+	−	−1	−1·2	−1	−1·2	−2	$-\infty$
$x =$	0°	30°	60°	90°	120°	150°	180° −0°	180° +0°	210°	240°	270°	300°	330°	360° −0°
COSEC −	+	2	1·2	1	1·2	2	+	−	−2	−1·2	−1	−1·2	−2	$-\infty$



The Cosecant Graph

Ex. 1. Show that $\sin 50° > \cos 50°$.

The angle is in the first quadrant where $\sin \theta$ increases from 0 to 1 and $\cos \theta$ decreases from 1 to 0. But at $45°$, $\sin 45° = \cos 45°$ because each of them is $\frac{1}{\sqrt{2}}$. After reaching $45°$, $\sin \theta$ increases while $\cos \theta$ decreases.

$\therefore \sin 50° > \cos 50°$.

Ex. 2. Determine whether $\sin A + \cos A$ is positive or negative when $A = 136°$.

The angle is in the second quadrant where $\sin A$ is positive and $\cos A$ is $-$ve. Also in this quadrant $\sin A$ decreases from 1 to 0 whereas $\cos A$ decreases from 0 to -1 and therefore $\cos A$ increases in magnitude. At $135°$ $\sin A$ and $\cos A$ are equal in magnitude (though opposite in sign). Therefore after that (*i. e.*, at $136°$) $\cos A$ is greater than $\sin A$ in magnitude and is negative, $\therefore \sin A + \cos A$ is $-$ve at $A = 136°$.

This can also be done as follows :—

$\sin 136° = \sin (180° - 44°) = \sin 44°$

$\cos 136° = \cos (180° - 44°) = -\cos 44°$.

Thus at $136°$, $\sin A + \cos A = \sin 44° - \cos 44°$. But it is easy to argue, as is done in Ex. 1, that $\cos 44° > \sin 44°$.

$\therefore \sin 44° - \cos 44°$ is negative.

EXERCISE XXV

1. Prove that

(*i*) $\tan A - \cot A$ is positive when $A = 53°$.

(*ii*) $\sin A - \cos B$ is not negative when A and B are between $45°$ and $90°$.

2. Prove that $\sin A + \cos A$ is positive if A lies between $45°$ and $135°$, but negative if A is between $135°$ and $225°$.

3. Trace the variations of $\sin \theta$ as θ varies from $-\pi$ to π and exhibit them by means of a graph. *(P. U. 1942 S.)*

4. Draw the graph of $y = \sin x$ as x varies from $0°$ to $180°$ and from the graph find out the values of x when (*i*) $\sin x = \cdot 3$, (*ii*) $\sin x = 6$. *(P. U.)*

5. Draw the graph of $y=\text{Cos } x$ when x varies from $-\pi$ to π and make use of the graph to solve the equations (*i*) $\text{Cos } x=\frac{4}{5}$. (*ii*) $\text{Cos } x=-\frac{3}{5}$.

6. With the same axes draw graphs of $y=\text{Sin } x$ and $y=\text{Cos } x$ for $0<x<2\pi$ and read off from your graph the roots of the equation $\text{Sin } x=\text{Cos } x$.

7. Use the graph of $y=\tan x$ to solve the equations (*i*) $\tan x=\frac{1}{2}$. (*ii*) $\tan x=-3$.

[**Hint** :—Here $\tan x=\frac{1}{2}$. Let $y=\tan x$ $\therefore y=\frac{1}{2}$.

Thus draw the graph $y=\tan x$ and read where $y=\frac{1}{2}$ cuts it.]

8. Draw the graph $y=\tan x$ for values of x lying between $0°$ and $180°$; show by means of this graph that $x=35$ is a solution of $x=50 \tan x$, where x is measured in degrees.

9. Trace the changes in (*i*) $\text{Sin } 2\theta$, (*ii*) $\tan 2\theta$, (*iii*) $\text{Sec } 2\theta$, as θ varies from $0°$ to $180°$ and exhibit them by means of graphs.

10. Trace the changes in $\text{Cos } \theta$ as θ varies from 0 to 2π and exhibit them graphically.

11. Solve graphically the equation $3 \text{ Sin } x=\text{Cos } x+2$ where x is acute.

[**Hint.** Draw the graphs of $y=3 \text{ Sin } x$ and $y=2+\text{Cos } x$ with the same axes.]

ANSWERS

EXERCISE I

1. $50\sqrt{3}$ ft. 2. $75\sqrt{3}$ yards 3. $125\sqrt{3}$ ft.

4. $30°$ 5. $11\frac{2}{3}$ ft. 6. $25\sqrt{3}$ ft.

7. 60 ft. 8. $100(\sqrt{3}+1)$ ft.

9. $20\sqrt{3}$, 20 yards 10. $125\sqrt{3}$ ft.

11. $100(\sqrt{3}-1)$ ft. 12. $300(\sqrt{3}+1)$ ft.

13. 200 ft. 14. 150 ft. 15. 200 ft.

EXERCISE II

2. (*i*) 0 (*ii*) 1 (*iii*) 1

6. (*i*) $-\tan A$ (*ii*) 1 (*iii*) -1 (*iv*) 1 (*v*) 3

9. $x=\tan A$

EXERCISE III

3. $\frac{24}{25}$; $-\frac{7}{25}$ 5. $-\frac{33}{65}$

6. $-\frac{33}{65}$ (if both are acute)

7. (*i*) Cos A (*ii*) Sin y (*iii*) $\frac{\sqrt{3}}{2}$ (*iv*) $\sqrt{3}$ Cos θ

EXERCISE IV

1. $-\frac{7}{25}$ 2. $\frac{47}{49}$ 3. $\pm\frac{120}{169}$ 4. $-\frac{5}{12}$

5. $\frac{4}{5}, -\frac{3}{5}$ 8. a 13. $\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}$

EXERCISE VI

1. $\frac{\sqrt{5}-1}{4}$; $\frac{\sqrt{10+2\sqrt{5}}}{4}$

3. $\sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}}$; $\sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}$

4. $\sqrt{\dfrac{\sqrt{2}-1}{2\sqrt{2}}}$; $\sqrt{\dfrac{\sqrt{2}+1}{2\sqrt{2}}}$; $\sqrt{\dfrac{\sqrt{2}-1}{\sqrt{2}+1}}$

5. $\frac{1}{2}$, $\dfrac{\sqrt{3}}{2}$ 7. $\dfrac{a}{b}$, $\dfrac{b}{a}$

9. $\frac{3}{7}$, $\dfrac{3}{\sqrt{58}}$, $\dfrac{7}{\sqrt{58}}$ 10. $2-\sqrt{3}$

EXERCISE VII

1. (*i*) $2 \text{ Sin } 2\theta \text{ Cos } \theta$ (*ii*) $2 \text{ Cos } 5\theta \text{ Sin } \theta$
(*iii*) $2 \text{ Cos } 5\theta \text{ Cos } 3\theta$ (*iv*) $2 \text{ Sin } 3\theta \text{ Sin } 2\theta$
(*v*) $2 \text{ Sin } 5\text{A} \text{ Sin } 2\text{A}$

2. (*i*) $\frac{1}{2}[\text{Cos } 60° + \text{Cos } 20°]$ (*ii*) $\frac{1}{2}[\text{Cos } 10\text{A} - \text{Cos } 12\text{A}]$
(*iii*) $\frac{1}{2}[\text{Sin } 10\text{A} - \text{Sin } 4\text{A}]$ (*iv*) $\text{Cos } 4\text{A} - \text{Cos } 10\text{A}$
(*v*) $\frac{1}{2}[\text{Cos } 4\text{A} - \text{Cos } 12\text{A}]$ (*vi*) $\frac{1}{2}[\text{Sin } 10\text{A} + \text{Sin } 4\text{A}]$

EXERCISE IX

1. (*i*) $x^2+y^2=a^2$ (*ii*) $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$

(*iii*) $\dfrac{a^2}{x^2}+\dfrac{b^2}{y^2}=1$ (*iv*) $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$

(*v*) $\dfrac{x^2}{a^2}-\dfrac{b^2}{y^2}=1$

2. (*i*) $x^2+y^2=2$ (*ii*) $(9x+7y)^2+(4x-5y)^2=(73)^2$
(*iii*) $q(q-p)=2$

3. $x^2+y^2-2xy \text{ Sin } (\alpha+\beta) = \text{Cos}^2 (\alpha+\beta)$

4. $(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}=(a^2-b^2)^{\frac{2}{3}}$ 5. $x^2+y^2=1$

6. $1-\dfrac{x}{a}=\dfrac{2y^2}{b^2}$ 7. $x(x^2-3)+2y=0$

EXERCISE X

1. $\dfrac{n\pi}{2}$ 2. $n\pi+(-1)^n \dfrac{\pi}{6}$ 3. $n\pi-(-1)^n \dfrac{\pi}{6}$

4. $\dfrac{n\pi}{3}+(-)^n \dfrac{\pi}{9}$ 5. $n\pi+(-1)^n \alpha$
when $\text{Sin } \alpha = p$

6. $n\pi+(-1)^n\alpha$ where Sin $\alpha=q$

7. $\frac{n\pi}{2}+(-1)^n\alpha$ 8. $n\pi+(-1)^n\alpha$

9. $2n\pi\pm\frac{\pi}{4}$ 10. $2n\pi\pm\frac{2\pi}{3}$

11. $\frac{n\pi}{2}\pm\frac{\pi}{24}$ 12. $2n\pi\pm\alpha$ when Cos $\alpha=p$

13. $\frac{2k\pi}{m\pm n}$ 14. $n\pi+\frac{\pi}{6}$

15. $n\pi+\frac{3\pi}{4}$ or $n\pi+\frac{\pi}{4}$

16. $n\pi+\alpha$ where tan $\alpha=q$

17. $n\pi+\alpha$ where tan $\alpha=p$

18. $\frac{(4n\pm1)\pi}{2(3\pm2)}$ 19. $\frac{(4k\pm1)\pi}{2(m\pm n)}$

20. $\theta=(2n+1)\frac{\pi}{14}$ 21. $\theta=n\pi\pm\frac{\pi}{6}$

22. $(2n+1)\frac{\pi}{16}$ 23. $n\pi\pm\frac{\pi}{4}$

24. $2n\pi\pm\frac{\pi}{3}$ 25. $2n\pi$

26. $n\pi\pm\frac{\pi}{3}$, $n\pi\pm\frac{\pi}{4}$

27. $2n\pi+\frac{7\pi}{6}$ 28. $2n\pi+\frac{\pi}{6}$

29. $(2n+1)\pi+\frac{\pi}{4}$ 30. $\left(n+\frac{m}{2}\right)\pi\pm\frac{\pi}{6}+(-1)^m\frac{\pi}{12}$

and $\left(\frac{m}{2}-n\right)\pi\pm\frac{\pi}{6}-(-1)^m\frac{\pi}{12}$

31. $x=\frac{1}{2}\left(6r\pi-4s\pi+\frac{\pi}{2}\pm\frac{2\pi}{3}\right)$

$y=\frac{1}{2}\left(6n\pi-4s\pi+\pi-\frac{\pi}{3}\right)$

32. $A=(m+n)\frac{\pi}{2}+\frac{5\pi}{24}$

$B=(l-m)\frac{\pi}{2}+\frac{\pi}{24}$

and $C=(l-n)\frac{\pi}{2}+\frac{\pi}{12}$ where l, m, n are integers.

EXERCISE XI

1. $2n\pi+\frac{\pi}{4}$

2. $2n\pi+\frac{7\pi}{12}$

3. $2n\pi+\frac{\pi}{3}$

4. $2n\pi+\frac{\pi}{2}$

5. $2n\pi+\frac{5\pi}{12}$

6. $2n\pi,\ 2n\pi+\frac{2\pi}{3}$

7. $2n\pi-\frac{\pi}{6}$

8. $(2n-1)\pi,\ 2n\pi+\frac{\pi}{3}$

9. $n\pi+\frac{\pi}{2},\ \frac{n\pi}{2}\pm\frac{\pi}{12}$

10. $\frac{n\pi}{2},\ 2n\pi\pm\frac{\pi}{3}$

11. $\frac{n\pi}{3},\ \frac{n\pi}{2}\pm\frac{\pi}{12}$

12. $\frac{n\pi}{2},\ 2n\pi,\ \frac{2}{3}n\pi$

13. $n\pi+\frac{\pi}{2},\ \frac{4n+\pi}{10}$

14. $2n\pi\pm\frac{2\pi}{3}$

15. $\frac{2p\pi}{m\pm n}$

16. $\frac{(4p+1)}{m\pm n}\cdot\frac{\pi}{2}$

17. $\tan\theta=\frac{(2n+1)\pm\sqrt{4n^2+4n-15}}{4}$

18. $\frac{p\pi+(-1)^p\frac{\pi}{2}}{m+(-1)^p n}$

19. $\frac{n\pi\pm\frac{\pi}{2}}{3\pm1}$

20. $\frac{n\pi+\frac{\pi}{2}}{4}$

21. $\frac{n\pi}{2}\pm\frac{\pi}{4}$

22. $(2n+1)\frac{\pi}{2}$; $n\pi \pm \frac{\pi}{3}$

23. $n\pi$ or $n\pi + (-1)^n\alpha$
where $\sin \alpha = 32$

24. $n\pi, n\pi - \varphi$ where $\tan \varphi = \frac{1}{2}$

25. $\frac{n\pi}{3}$; $n\pi \pm \alpha$ where $\tan \alpha = \frac{1}{\sqrt{2}}$

EXERCISE XVI

1. (i) $\sqrt{2}, \log_2 1{\cdot}414 = {\cdot}5$ (ii) $\frac{1}{3}, \log_{27} {\cdot}3 = -{\cdot}5$
(iii) $8, \log_{16} 8 = {\cdot}75$ (iv) ${\cdot}25, \log_{256} {\cdot}25 = -{\cdot}25$

3. (i) 3 (ii) −1 (iii) −1 (iv) 3

5. (i) 0 (ii) $8\ (\log_a 3 - \log_a 2) - 5 \log_a 7$
(iii) $\frac{15}{2} \log_a 2 + 3 \log_a 5 + 4 \log_a 13$ (iv) $-\frac{9}{4} \log_a 7$

EXERCISE XVII

(i) 0 (ii) 2 (iii) 5 (iv) −1
(v) −3 (vi) −6 (vii) −1 (viii) −2
(ix) 1 (x) 5

EXERCISE XVIII

1. (i) 3·6352 (ii) 1·1038 (iii) $\bar{3}{\cdot}6352$
(iv) 2·1038 (v) $\bar{1}{\cdot}6352$ (vi) $\bar{1}{\cdot}1038$
(vii) 5·1038

2. (i) 31 (ii) 110 (iii) 140 (iv) 126

3. (i) 11 (ii) 209 (iii) 176 (iv) 39

EXERCISE XIX

2. (i) −1·256 app. (ii) ·03 (iii) 107·7 app.

3. 7 4. 21 5. ·3948330 6. 9·59573

7. 9·9604747 8. 10·6132960 9. 17°27′ 43″

10. 16·43 years.

EXERCISE XX

1. $A=15°$, $b=50\,(2-\sqrt{3})$. $c=50\,(\sqrt{6}+\sqrt{2})$
2. $5°-44'-20''$;
3. $A=49°-20'-30''$; $B=40°-39'-30''$.
4. $36°-26'-7{\cdot}7''$
5. $26°-33'-54''$; $63°-26-6''$; $60\sqrt{5}$ ft.

EXERCISE XXI

1. $b=61{\cdot}51$, $c=32{\cdot}51$; $A=59°\ 30'$
2. $b=25{\cdot}07$ $c=26{\cdot}55$
3. $b=237$, $c=1{\cdot}581$, $A=66°\ 20'$
4. $A-42°\ 54'$, $b=25{\cdot}07$ $c=25.56$
5. $87°\ 8'$; $a=298$, $b=14{\cdot}35$

EXERCISE XXII

1. $A=32°-12'$; $B=46°-12'$; $C=101-36'$
2. $A=60°\ 10'$ $A=49°\ 28'$; $B=58°\ 46'$
4. $A=20°-56'$; $B=26°-30'$; $C=132°-34'$
5. $A=60°-10'$; $B=28°-8'$; $C=91°-42'$
6. Area$=551300$ sq. ft. ; $r=148{\cdot}68$ ft.
7. $A=90°-4'$; $B=48°-6'$; $C=40°-50'$

EXERCISE XXIII

1. $B=41°-22'$
2. $B=97°-30'$, $C=35°—30'$, $a=18{\cdot}51$
3. $B=73°-35'$, $C=39°-45'$
   $a=226{\cdot}9$
4. $B=92°-41'$, $C=54°-49'$,
   $a=5{\cdot}917$
5. $B=118°-37'$; $C=31°-45'$
   $a=20{\cdot}95$
6. $71°-44'-29''$; $48°-15'-31''$
7. $B=78°-48'-52''$; $C=56°-41'-''8$
8. $B=56°-19'-46''$; $C=63°-40'-14''$

EXERCISE XXIV

1. 30° ; **2.** (*i*) Two Solutions : $b_1=60{\cdot}3893$ $B_1=8^\circ-41'$; $B_2=111^\circ-19'$, $C_1=141^\circ-19'$, $C_2=38^\circ-41'$
(*ii*) Only one solution : $C=18^\circ-12'-40''$, $B=131^\circ-47'-20''$

(*iii*) Only one solution : $C=90^\circ$, $B=60^\circ$

3. $17{\cdot}1$ or $3{\cdot}68$ **4.** $39^\circ-35'-10''$; $28^\circ-20'-50''$

5. $B_1=58^\circ-56'-56''$. $B_2=121^\circ-3'-4''$
$C_1=87^\circ-48'-7''$, $C_2=25^\circ-41'-53''$

JAMMU AND KASHMIR UNIVERSITY PAPERS

K. U. 1957

1. (*a*) Define a radian, show that it is a constant angle and express it in sexagesimal measure correct to the nearest second.

What is the difference between π and π radians ?

(*b*) If G, D, C be the number of grades, degrees and radians in any angle, prove that

$$\frac{D}{9}=\frac{G}{10}=\frac{20C}{\pi}$$

2. (*a*) Prove that $\text{Sec}^2\theta=1+\tan^2\theta$ where θ is any angle.

(*b*) Prove the identity $(\text{Sin } x+\text{Sec } x)^2+(\text{Cosec } x+\text{Cos } x)^2=(1+\text{Sec } x \text{ Cosec } x)^2$.

(*c*) Two posts of the same height stand on either side of a road 120 ft. wide ; at a point in the road between the posts, the elevations of the tops of the pillars are 60° and 30°. Find height of the posts and the position of the point.

3. (*a*) Prove that for all values of θ, $\tan(\pi+\theta)=\tan\theta$.

(*b*) Draw the graph of $\tan\theta$ for $0\leqslant\theta\leqslant2\pi$ and find from the graph the values of θ which satisfy the equation $\tan\theta=\text{Cot}\theta$.

(*c*) Prove that $\tan\theta\tan\left(\frac{\pi}{2}+\theta\right)+1=0$

4. (*a*) If $\alpha+\beta+\gamma=\frac{\pi}{2}$, prove that

$$\tan\alpha\tan\beta+\tan\beta\tan\gamma+\tan\gamma\tan\alpha=1.$$

(*b*) Find the circular functions of 18°.

(*c*) Prove that $\text{Cos } 20° \text{ Cos } 40° \text{ Cos } 60° \text{ Cos } 80°=\frac{1}{16}$.

5. (*a*) To prove that in any $\triangle ABC$, $\dfrac{\tan\frac{B-C}{2}}{\tan\frac{B+C}{2}}=\dfrac{b-c}{b+c}$

(*b*) If a, b, c are in H. P., prove that $\text{Sin}^2 \frac{A}{2}$,

$\text{Sin}^2 \frac{B}{2}$, $\text{Sin}^2 \frac{C}{2}$ are also in H. P.

(*c*) Solve the equation $\text{Sin } 4\theta = \text{Sin } \theta$.

6. (*a*) If $a = 182{\cdot}5$, $b = 82{\cdot}5$, $A = 72^\circ\ 15'$, solve the triangle.

(*b*) Prove the formula $R = \frac{a}{2 \text{ Sin } A} = \frac{b}{2 \text{ Sin } B} = \frac{c}{2 \text{ Sin } C}$,

where R is the circumradius of a triangle ABC.

K. U. 1958

1. (*a*) Show that the length of an arc subtending an angle θ radians at the centre of a circle of radius r, is $r\theta$.

(*b*) A pendulum 8 ft. long oscillates through an angle of 9°; what is the length of the path its extremity describes between the extreme positions ?

(*c*) The angles of a quadrilateral are in A. P. and the greatest is double the least; express the least angle in degrees and grades.

2. (*a*) Construct angles between 0° and 360° whose tangent is $\frac{2}{3}$ and find their Secants and Cosecants.

(*b*) Prove that $(\tan \theta + \text{Sec } \theta)^2 = \frac{\text{Cosec } \theta + 1}{\text{Cosec } \theta - 1}$

(*c*) In a cyclic quadrilateral ABCD, show that :
$\text{Cos } A + \text{Cos } C = 0$ and $\text{Cos } B + \text{Cos } D = 0$.

3. (*a*) Two men A and B, 1360 yds. apart observe an aeroplane C at the same instant and find the respective angles of elevations to be 45° and 60°. If the plane ABC is vertical, find the height of the aeroplane.

(*b*) Draw the graphs of $\tan \theta$ and $\text{Cot } \theta$ between $\theta = 0$ and $\theta = \pi$ and from your graph find the values of θ which satisfy $\tan \theta = \text{Cot } \theta$.

4. (*a*) Prove that $\text{Cos }(A+B) \text{ Cos }(A-B) = \text{Cos}^2 A - \text{Sin}^2 B$

(*b*) Prove that $\text{Sin } 70^\circ - \text{Cos } 80^\circ = \text{Cos } 40^\circ$.

(c) Prove that, if $A+B+C=180^\circ$, then

$$\frac{\text{Sin } 2A+\text{Sin } 2B+\text{Sin } 2C}{\text{Sin } A+\text{Sin } B+\text{Sin } C}=8 \text{ Sin } \frac{A}{2} \text{ Sin } \frac{B}{2} \text{ Sin } \frac{C}{2}$$

5. (a) Solve $\text{Sin } \theta+\text{Sin } 2\theta+\text{Sin } 3\theta=0$.

(b) In any triangle ABC, prove that $\text{Cos } \frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}$

where $a+b+c=2s$.

(c) Prove that $\text{Sin } A+\text{Sin } B+\text{Sin } C=\frac{s}{R}$ in any $\triangle$ ABC where R is the Circum-radius and $a+b+c=2s$

6. (a) Given $\log 2=\cdot30103$, find the number of digits 2^{24}.

(b) If $A=50^\circ$, $b=1071$, $a=873$; find to the nearest second, angle B. Given $\log 1\cdot071=\cdot029789$, L Sin $50^\circ=9\cdot884254$, L Sin $70^\circ=9\cdot972986$, L Sin $70^\circ\ 1'=9\cdot973032$, $\log 8\cdot73=\cdot9410$14.

K. U. 1959

1. (a) Prove that the radian is a constant angle.

(b) Show that $\frac{\tan A+\text{Sec } A-1}{\tan A-\text{Sec } A+1}=\frac{1+\text{Sin } A}{\text{Cos } A}$

2. (a) Trace the changes in the sign and magnitude of the trigonometrical ratios of an angle as the angle increases from 0° to 360°.

(b) Find a solution of the equation.
$3 \tan \theta+\text{Cot } \theta=5 \text{ Cosec } \theta$.

3. (a) Prove geometrically that
$\text{Cos } (A-B)=\text{Cos } A \text{ Cos } B+\text{Sin } A \text{ Sin } B$.

(b) Find the expansion of Cos 3 A.

4. (a) In a $\triangle$ABC if $a \text{ Cos}^2 \frac{C}{2}+c \text{ Cos}^2 \frac{A}{2}=\frac{3b}{2}$, show that the sides of the triangle are in A. P.

(b) Prove that
$\log_a m=\log_b m \times \log_a b$

5. (*a*) If $A+B+C=180^0$, Prove that

$$\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}=1-2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

(*b*) In a $\triangle$ ABC prove that

$$R=\frac{a}{2\sin A}.$$

K.U. 1960

1. (*a*) Prove that $(1+\text{Cot}\,A+\tan A)(\sin A-\cos A)$

$$=\frac{\sec A}{\text{Cosec}^2 A}-\frac{\text{Cosec}\,A}{\sec^2 A}$$

(*b*) From the top of a cliff, 200 feet high, the angles of depression of the top and bottom of a tower are observed to be 30° and 60° respectively ; find the height of the tower.

2. (*a*) Prove geometrically that $\sin(A+B)=\sin A\cos B+\cos A\sin B$.

(*b*) Show that $\dfrac{\sin^2 A-\sin^2 B}{\sin A\cos A-\sin B\cos B}=\tan(A+B)$

3. (*a*) If $A+B+C=180^\circ$ then show that
$\sin^2 A+\sin^2 B+\sin^2 C=2+2\cos A\cos B\cos C$.

(*b*) Solve the equation : $\sin\theta+\sin 7\theta=\sin 4\theta$.

4. (*a*) Prove that (*i*) $\log_a\left(\frac{m}{n}\right)=\log_a{}^m-\log_a{}^n$,

(*ii*) $\log_a(m^n)=n\log_a{}^{nm}$.

(*b*) Show that in any $\triangle$ ABC, $\cos C=\dfrac{a^2+b^2-c^2}{2ab}$

5. If $b=\sqrt{3}$, $c=1$ and $A=30^\circ$, then solve the $\triangle$ABC.

(*b*) If r be the radius of the incircle of the triangle ABC, then Show that $r=\dfrac{\triangle}{s}$, where $\triangle$ and s denote respectively the area and the semi-perimeter of the triangle ABC.

K.U. 1961

1. (*a*) Prove that

$$\text{Sin}A=\frac{2\tan A/2}{1+\tan^2 A/2} \text{ and } \text{Cos } A=\frac{1-\tan^2 A/2}{1+\tan^2 A/2}.$$

(*b*) The shadow of a tower standing on a level plane is found to be 60 feet longer when the sun's altitude is 30° than when it is 45°. Prove that the height of the tower is $30\ (1+\sqrt{3}\ )$ feet.

2. (*a*) Prove that $\text{Sin}\ (A+B)\ \text{Sin}\ (A-B)=\text{Sin}^2 A-\text{Sin}^2 B$
and $\text{Cos}\ (A+B)\ \text{Cos}\ (A-B)=\text{Cos}^2 A-\text{Sin}^2 B$

(*b*) Show that $1+\tan A\ \tan A/2=\tan A\ \text{Cot}\ A/2-1$ $=\text{Sec}\ A$.

3. (*a*) If $A+B+C=180^\circ$, prove that
$\tan A/2\ \tan B/2+\tan B/2\ \tan C/2+\tan C/2\ \tan A/2=1$

(*b*) Solve the equation $\text{Sin}\ \theta+\text{Sin}\ 5\theta=\text{Sin}\ 3\theta$.

4. (*a*) Having given $\log 3=\cdot4771213$, find the number of digits in 3^{62}.

(*b*) In any $\triangle ABC$, prove that

$$\frac{a\ \text{Sin}\ (B-C)}{b^2-c^2}=\frac{b\ \text{Sin}\ (C-A)}{c^2-a^2}=\frac{c\ \text{Sin}\ (A-B)}{a^2-b^2}.$$

5. (*a*) Show that in any triangle ABC,

$$\tan\ (B-C)/2=\frac{b-c}{b-c}\ \text{Cot}\ A/2.$$

(*b*) If R and r denote respectively the radii of the circumcircle and the incircle of any triangle ABC, prove that $1/bc+1/ca+1/ab=1/2Rr$.

Higher Secondary 1961 (J & K. University)

Note :—Do questions worth 44 marks. Complete questions are to be attempted].

1. (*a*) Prove that a radian is an angle of constant magnitude.

(*b*) Express $2{\cdot}2$ radian in the Sexagesimal and Centesimal Systems.

2. (*a*) Express all the circular functions of θ in terms of Cos θ.

(*b*) Given that $\tan\theta=\frac{3}{5}$, when θ lies in third quadrant, find the other circular functions of θ.

Or

Eliminate θ from
$$a\operatorname{Cos}\theta+b\operatorname{Sin}\theta+c=0$$
$$a_1\operatorname{Cos}\theta+b_1\operatorname{Sin}\theta+c_1=0$$

3. (*a*) Prove that the logarithm of the product of two factors is equal to the sum of the logarithms of the factors.

(*b*) If $a^2+b^2=7ab$, then $\log\left(\frac{a+b}{3}\right)$

$=\frac{1}{2}(\log a+\log b)$

4. (*a*) Solve the equation $5^{7-4x}=2^{x+3}$, given that Log $2=\cdot3010$.

(*b*) Given that Log $2=\cdot3010$, find the position of the first significant figure in 2^{-35}

5. (*a*) AD is the bisector of $\angle$A of the $\triangle$ABC, meeting BC in D. Prove that

$$BD=\frac{a\operatorname{Sin}C}{\operatorname{Sin}C+\operatorname{Sin}B},\ CD=\frac{a\operatorname{Sin}B}{\operatorname{Sin}C+\operatorname{Sin}B}$$

(*b*) In a $\triangle$ ABC, if $\frac{\operatorname{Cos}A}{a}=\frac{\operatorname{Cos}B}{b}=\frac{\operatorname{Cos}C}{c}$,

Prove that the triangle is equilateral.

6. A circle with radius R passes through the vertices A, B and C of the $\triangle$ ABC. Find that the

$$\triangle ABC=\frac{\text{Production}}{4R}$$

Or

At a point 200 ft. from the base of a tower which stands on a horizontal plane, the angle of elevation of the top is 60°. Find the length of the tower.

TABLES OF LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170						5	9	13	17	21	26	30	34	38
						0212	0253	0294	0334	0374	4	8	12	16	20	24	28	32	36
11	0414	0453	0492	0531	0569						4	8	12	16	20	23	27	31	35
						0607	0645	0682	0719	0755	4	7	11	15	18	22	26	29	33
12	0792	0828	0864	0899	0934						3	7	11	14	18	21	25	28	32
						0969	1004	1038	1072	1106	3	7	10	14	17	20	24	27	31
13	1139	1173	1206	1239	1271						3	6	10	13	16	19	23	26	29
						1303	1335	1367	1399	1430	3	7	10	13	16	19	22	25	29
14	1461	1492	1523	1553	1584						3	6	9	12	15	19	22	25	28
						1614	1644	1673	1703	1732	3	6	9	12	15	17	20	23	26
15	1761	1790	1818	1847	1875						3	6	9	11	14	17	20	23	26
						1903	1931	1959	1987	2014	3	6	8	11	14	17	19	22	25
16	2041	2068	2095	2122	2148						3	6	8	11	14	16	19	22	24
						2175	2201	2227	2253	2279	3	5	8	10	13	16	18	21	23
17	2304	2330	2355	2380	2405						3	5	8	10	13	15	18	20	23
						2430	2455	2480	2504	2529	3	5	8	10	12	15	17	20	22
18	2553	2577	2601	2625	2648						2	5	7	9	12	14	17	19	21
						2672	2695	2718	2742	2765	2	4	7	9	11	14	15	18	21
19	2788	2810	2833	2856	2878						2	4	7	9	11	13	16	18	20
						2900	2923	2945	2967	2989	2	4	6	8	11	13	15	17	19
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	11
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	57[illegible]	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	[illegible]
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	[illegible]
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8

	0	1	2	3	4	5	6	7	8	9	123	456	789
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1 2 3	3 4 5	6 7 8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1 2 3	3 4 5	6 7 8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1 2 2	3 4 5	6 7 7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1 2 2	3 4 5	6 6 7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1 2 2	3 4 5	6 6 7
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1 2 2	3 4 5	5 6 7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1 2 2	3 4 5	5 6 7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1 2 2	3 4 5	5 6 7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1 1 2	3 4 4	5 6 7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1 1 2	3 4 4	5 6 7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1 1 2	3 4 4	5 6 6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1 1 2	3 4 4	5 6 6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1 1 2	3 3 4	5 6 6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1 1 2	3 3 4	5 5 6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1 1 2	3 3 4	5 5 6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1 1 2	3 3 4	5 5 6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1 1 2	3 3 4	5 5 6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1 1 2	3 3 4	5 5 6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1 1 2	3 3 4	4 5 6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1 1 2	2 3 4	4 5 6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1 1 2	2 3 4	4 5 6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1 1 2	2 3 4	4 5 5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1 1 2	2 3 4	4 5 5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1 1 2	2 3 4	4 5 5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1 1 2	2 3 4	4 5 5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1 1 2	2 3 3	4 5 5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1 1 2	2 3 3	4 5 5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1 1 2	2 3 3	4 4 5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1 1 2	2 3 3	4 4 5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1 1 2	2 3 3	4 4 5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1 1 2	2 3 3	4 4 5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1 1 2	2 3 3	4 4 5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1 1 2	2 3 3	4 4 5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1 1 2	2 3 3	4 4 5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1 1 2	2 3 3	4 4 5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1 1 2	2 3 3	4 4 5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1 1 2	2 3 3	4 4 5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0 1 1	2 2 3	3 4 4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0 1 1	2 2 3	3 4 4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0 1 1	2 2 3	3 4 4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0 1 1	2 2 3	3 4 4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0 1 1	2 2 3	3 4 4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0 1 1	2 2 3	3 4 4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0 1 1	2 2 3	3 4 4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0 1 1	2 2 3	3 4 4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0 1 1	2 2 3	3 4 4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0 1 1	2 2 3	3 4 4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0 1 1	2 2 3	3 4 4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0 1 1	2 2 3	3 4 4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0 1 1	2 2 3	3 3 4

	0	1	2	3	4	5	6	7	8	9	123	456	789
·00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0 0 1	1 1 1	2 2 2
·01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0 0 1	1 1 1	2 2 2
·02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0 0 1	1 1 1	2 2 2
·03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0 0 1	1 1 1	2 2 2
·04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0 1 1	1 1 2	2 2 2
·05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0 1 1	1 1 2	2 2 2
·06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0 1 1	1 1 2	2 2 2
·07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0 1 1	1 1 2	2 2 2
·08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0 1 1	1 1 2	2 2 3
·09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0 1 1	1 1 2	2 2 3
·10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0 1 1	1 1 2	2 2 3
·11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0 1 1	1 2 2	2 2 3
·12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0 1 1	1 2 2	2 2 3
·13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0 1 1	1 2 2	2 3 3
·14	[illegible]	1384	1387	1390	1393	1396	1400	1403	1406	1409	0 1 1	1 2 2	2 3 3
·15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0 1 1	1 2 2	2 3 3
·16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0 1 1	1 2 2	2 3 3
·17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0 1 1	1 2 2	2 3 3
·18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0 1 1	1 2 2	2 3 3
·19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0 1 1	1 2 2	3 3 3
·20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0 1 1	1 2 2	3 3 3
·21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0 1 1	2 2 2	3 3 3
·22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0 1 1	2 2 2	3 3 3
·23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0 1 1	2 2 2	3 3 4
·24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0 1 1	2 2 2	3 3 4
·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0 1 1	2 2 2	3 3 4
·26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0 1 1	2 2 3	3 3 4
·27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0 1 1	2 2 3	3 3 4
·28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0 1 1	2 2 3	3 4 4
·29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0 1 1	2 2 3	3 4 4
·30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0 1 1	2 2 3	3 4 4
·31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0 1 1	2 2 3	3 4 4
·32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0 1 1	2 2 3	3 4 4
·33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0 1 1	2 2 3	3 4 4
·34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1 1 2	2 3 3	4 4 5
·35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1 1 2	2 3 3	4 4 5
·36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1 1 2	2 3 3	4 4 5
·37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1 1 2	2 3 3	4 4 5
·38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1 1 2	2 3 3	4 4 5
·39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1 1 2	2 3 3	4 5 5
·40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1 1 2	2 3 4	4 5 5
·41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1 1 2	2 3 4	4 5 5
·42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1 1 2	2 3 4	4 5 6
·43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1 1 2	2 3 4	4 5 6
·44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1 1 2	3 3 4	4 5 6
·45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1 1 2	3 3 4	5 5 6
·46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1 1 2	3 3 4	5 5 6
·47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1 1 2	3 3 4	5 5 6
·48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1 1 2	3 3 4	5 5 6
·49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1 1 2	3 4 4	5 6 6

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
·51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
·52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
·53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
·54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
·55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
·56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
·57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
·58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
·59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
·60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
·61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
·62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
·63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
·64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
·65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
·66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
·67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
·68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
·69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
·70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
·71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
·72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
·73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
·74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
·75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
·76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
·77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
·78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
·79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
·80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
·81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
·82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
·83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
·84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
·85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
·86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
·87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
·88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
·89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
·90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
·91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
·92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
·93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
·94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
·95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
·96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
·97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
·98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
·99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

NATURAL SINES

Degrees	0′ 0°·0	6′ 0°·1	12′ 0°·2	18′ 0°·3	24′ 0°·4	30′ 0°·5	36′ 0°·6	42′ 0°·7	48′ 0°·8	54′ 0°·9	Mean Differences 1	2	3	4	5
0	·0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	3	6	9	12	15
1	·0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	3	6	9	12	15
2	·0349	0366	0384	0401	0419	0436	0454	0471	0488	0506	3	6	9	12	15
3	·0523	0541	0558	0576	0593	0610	0628	0645	0663	0680	3	6	9	12	15
4	·0698	0715	0732	0750	0767	0785	0802	0819	0837	0854	3	6	9	12	15
5	·0872	0889	0906	0924	0941	0958	0976	0993	1011	1028	3	6	9	12	14
6	·1045	1063	1080	1097	1115	1132	1149	1167	1184	1201	3	6	9	12	14
7	·1219	1236	1253	1271	1288	1305	1323	1340	1357	1374	3	6	9	12	14
8	·1392	1409	1426	1444	1461	1478	1495	1513	1530	1547	3	6	9	12	14
9	·1564	1582	1599	1616	1633	1650	1668	1685	1702	1719	3	6	9	12	14
10	·1736	1754	1771	1788	1805	1822	1840	1857	1874	1891	3	6	9	12	14
11	·1908	1925	1942	1959	1977	1994	2011	2028	2045	2062	3	6	9	11	14
12	·2079	2096	2113	2130	2147	2164	2181	2198	2215	2232	3	6	9	11	14
13	·2250	2267	2284	2300	2317	2334	2351	2368	2385	2402	3	6	8	11	14
14	·2419	2436	2453	2470	2487	2504	2521	2538	2554	2571	3	6	8	11	14
15	·2588	2605	2622	2639	2656	2672	2689	2706	2723	2740	3	6	8	11	14
16	·2756	2773	2790	2807	2823	2840	2857	2874	2890	2907	3	6	8	11	14
17	·2924	2940	2957	2974	2990	3007	3024	3040	3057	3074	3	6	8	11	14
18	·3090	3107	3123	3140	3156	3173	3190	3206	3223	3239	3	6	8	11	14
19	·3256	3272	3289	3305	3322	3338	3355	3371	3387	3404	3	5	8	11	14
20	·3420	3437	3453	3469	3486	3502	3518	3535	3551	3567	3	5	8	11	14
21	·3584	3600	3616	3633	3649	3665	3681	3697	3714	3730	3	5	8	11	14
22	·3746	3762	3778	3795	3811	3827	3843	3859	3875	3891	3	5	8	11	14
23	·3907	3923	3939	3955	3971	3987	4003	4019	4035	4051	3	5	8	11	14
24	·4067	4083	4099	4115	4131	4147	4163	4179	4195	4210	3	5	8	11	13
25	·4226	4242	4258	4274	4289	4305	4321	4337	4352	4368	3	5	8	11	13
26	·4384	4399	4415	4431	4446	4462	4478	4493	4509	4524	3	5	8	10	13
27	·4540	4555	4571	4586	4602	4617	4633	4648	4664	4679	3	5	8	10	13
28	·4695	4710	4726	4741	4756	4772	4787	4802	4818	4833	3	5	8	10	13
29	·4848	4863	4879	4894	4909	4924	4939	4955	4970	4985	3	5	8	10	13
30	·5000	5015	5030	5045	5060	5075	5090	5105	5120	5135	3	5	8	10	13
31	·5150	5165	5180	5195	5210	5225	5240	5255	5270	5284	2	5	7	10	12
32	·5299	5314	5329	5344	5358	5373	5388	5402	5417	5432	2	5	7	10	12
33	·5446	5461	5476	5490	5505	5519	5534	5548	5563	5577	2	5	7	10	12
34	·5592	5606	5621	5635	5650	5664	5678	5693	5707	5721	2	5	7	10	12
35	·5736	5750	5764	5779	5793	5807	5821	5835	5850	5864	2	5	7	10	12
36	·5878	5892	5906	5920	5934	5948	5962	5976	5990	6004	2	5	7	9	12
37	·6018	6032	6046	6060	6074	6088	6101	6115	6129	6143	2	5	7	9	12
38	·6157	6170	6184	6198	6211	6225	6239	6252	6266	6280	2	5	7	9	11
39	·6293	6307	6320	6334	6347	6361	6374	6388	6401	6414	2	4	7	9	11
40	·6428	6441	6455	6468	6481	6494	6508	6521	6534	6547	2	4	7	9	11
41	·6561	6574	6587	6600	6613	6626	6639	6652	6665	6678	2	4	7	9	11
42	·6691	6704	6717	6730	6743	6756	6769	6782	6794	6807	2	4	6	9	11
43	·6820	6833	6845	6858	6871	6884	6896	6909	6921	6934	2	4	6	8	11
44	·6947	6959	6972	6984	6997	7009	7022	7034	7046	7059	2	4	6	8	10

NATURAL SINES

Degrees	0′ 0°·0	6′ 0°·1	12′ 0°·2	18′ 0°·3	24′ 0°·4	30′ 0°·5	36′ 0°·6	42′ 0°·7	48′ 0°·8	54′ 0°·9	Mean Differences 1	2	3	4	5
45	·7071	7083	7096	7108	7120	7133	7145	7157	7169	7181	2	4	6	8	10
46	·7193	7206	7218	7230	7242	7254	7266	7278	7290	7302	2	4	6	8	10
47	·7314	7325	7337	7349	7361	7373	7385	7396	7408	7420	2	4	6	8	10
48	·7431	7443	7455	7466	7478	7490	7501	7513	7524	7536	2	4	6	8	10
49	·7547	7558	7570	7581	7593	7604	7615	7627	7638	7649	2	4	6	8	9
50	·7660	7672	7683	7694	7705	7716	7727	7738	7749	7760	2	4	6	7	9
51	·7771	7782	7793	7804	7815	7826	7837	7848	7859	7869	2	4	5	7	9
52	·7880	7891	7902	7912	7923	7934	7944	7955	7965	7976	2	4	5	7	9
53	·7986	7997	8007	8018	8028	8039	8049	8059	8070	8080	2	3	5	7	9
54	·8090	8100	8111	8121	8131	8141	8151	8161	8171	8181	2	3	5	7	8
55	·8192	8202	8211	8221	8231	8241	8251	8261	8271	8281	2	3	5	7	8
56	·8290	8300	8310	8320	8329	8339	8348	8358	8368	8377	2	3	5	6	8
57	·8387	8396	8406	8415	8425	8434	8443	8453	8462	8471	2	3	5	6	8
58	·8480	8490	8499	8508	8517	8526	8536	8545	8554	8563	2	3	5	6	8
59	·8572	8581	8590	8599	8607	8616	8625	8634	8643	8652	1	3	4	6	7
60	·8660	8669	8678	8686	8695	8704	8712	8721	8729	8738	1	3	4	6	7
61	·8746	8755	8763	8771	8780	8788	8796	8805	8813	8821	1	3	4	6	7
62	·8829	8838	8846	8854	8862	8870	8878	8886	8894	8902	1	3	4	5	7
63	·8910	8918	8926	8934	8942	8949	8957	8965	8973	8980	1	3	4	5	6
64	·8988	8996	9003	9011	9018	9026	9033	9041	9048	9056	1	3	4	5	6
65	·9063	9070	9078	9085	9092	9100	9107	9114	9121	9128	1	2	4	5	6
66	·9135	9143	9150	9157	9164	9171	9178	9184	9191	9198	1	2	3	5	6
67	·9205	9212	9219	9225	9232	9239	9245	9252	9259	9265	1	2	3	4	6
68	·9272	9278	9285	9291	9298	9304	9311	9317	9323	9330	1	2	3	4	5
69	·9336	9342	9348	9354	9361	9367	9373	9379	9385	9391	1	2	3	4	5
70	·9397	9403	9409	9415	9421	9426	9432	9438	9444	9449	1	2	3	4	5
71	·9455	9461	9466	9472	9478	9483	9489	9494	9500	9505	1	2	3	4	5
72	·9511	9516	9521	9527	9532	9537	9542	9548	9553	9558	1	2	3	3	4
73	·9563	9568	9573	9578	9583	9588	9593	9598	9603	9608	1	2	2	3	4
74	·9613	9617	9622	9627	9632	9636	9641	9646	9650	9655	1	2	2	3	4
75	·9659	9664	9668	9673	9677	9681	9686	9690	9694	9699	1	1	2	3	4
76	·9703	9707	9711	9715	9720	9724	9728	9732	9736	9740	1	1	2	3	3
77	·9744	9748	9751	9755	9759	9763	9767	9770	9774	9778	1	1	2	3	3
78	·9781	9785	9789	9792	9796	9799	9803	9806	9810	9813	1	1	2	2	3
79	·9816	9820	9823	9826	9829	9833	9836	9839	9842	9845	1	1	2	2	3
80	·9848	9851	9854	9857	9860	9863	9866	9869	9871	9874	0	1	1	2	2
81	·9877	9880	9882	9885	9888	9890	9893	9895	9898	9900	0	1	1	2	2
82	·9903	9905	9907	9910	9912	9914	9917	9919	9921	9923	0	1	1	2	2
83	·9925	9928	9930	9932	9934	9936	9938	9940	9942	9943	0	1	1	1	2
84	·9945	9947	9949	9951	9952	9954	9956	9957	9959	9960	0	1	1	1	2
85	·9962	9963	9965	9966	9968	9969	9971	9972	9973	9974	0	0	1	1	1
86	·9976	9977	9978	9979	9980	9981	9982	9983	9984	9985	0	0	1	1	1
87	·9986	9987	9988	9989	9990	9990	9991	9992	9993	9993	0	0	0	1	1
88	·9994	9995	9995	9996	9996	9997	9997	9997	9998	9998	0	0	0	0	0
89	·9998	9999	9999	9999	9999	1·000	1·000	1·000	1·000	1·000	0	0	0	0	0
90	1·000														

NATURAL COSINES

[Numbers in difference columns to be subtracted, not added.]

Degrees	0′ 0°·0	6′ 0°·1	12′ 0°·2	18′ 0°·3	24′ 0°·4	30′ 0°·5	36′ 0°·6	42′ 0°·7	48′ 0°·8	54′ 0°·9	Mean Differences 1	2	3	4	5
0	1·000	1·000	1·000	1·000	1·000	1·000	·9999	9999	9999	9999	0	0	0	0	0
1	·9998	9998	9998	9997	9997	9997	9996	9996	9995	9995	0	0	0	0	0
2	·9994	9993	9993	9992	9991	9990	9990	9989	9988	9987	0	0	0	1	1
3	·9986	9985	9984	9983	9982	9981	9980	9979	9978	9977	0	0	1	1	1
4	·9976	9974	9973	9972	9971	9969	9968	9966	9965	9963	0	0	1	1	1
5	·9962	9960	9959	9957	9956	9954	9952	9951	9949	9947	0	1	1	1	2
6	·9945	9943	9942	9940	9938	9936	9934	9932	9930	9928	0	1	1	1	2
7	·9925	9923	9921	9919	9917	9914	9912	9910	9907	9905	0	1	1	2	2
8	·9903	9900	9898	9895	9893	9890	9888	9885	9882	9880	0	1	1	2	2
9	·9877	9874	9871	9869	9866	9863	9860	9857	9854	9851	0	1	1	2	2
10	·9848	9845	9842	9839	9836	9833	9829	9826	9823	9820	1	1	2	2	3
11	·9816	9813	9810	9806	9803	9799	9796	9792	9789	9785	1	1	2	2	3
12	·9781	9778	9774	9770	9767	9763	9759	9755	9751	9748	1	1	2	3	3
13	·9744	9740	9736	9732	9728	9724	9720	9715	9711	9707	1	1	2	3	3
14	·9703	9699	9694	9690	9686	9681	9677	9673	9668	9664	1	1	2	3	4
15	·9659	9655	9650	9646	9641	9636	9632	9627	9622	9617	1	2	2	3	4
16	·9613	9608	9603	9598	9593	9588	9583	9578	9573	9568	1	2	2	3	4
17	·9563	9558	9553	9548	9542	9537	9532	9527	9521	9516	1	2	3	3	4
18	·9511	9505	9500	9494	9489	9483	9478	9472	9466	9461	1	2	3	4	5
19	·9455	9449	9444	9438	9432	9426	9421	9415	9409	9403	1	2	3	4	5
20	·9397	9391	9385	9379	9373	9367	9361	9354	9348	9342	1	2	3	4	5
21	·9336	9330	9323	9317	9311	9304	9298	9291	9285	9278	1	2	3	4	5
22	·9272	9265	9259	9252	9245	9239	9232	9225	9219	9212	1	2	3	4	6
23	·9205	9198	9191	9184	9178	9171	9164	9157	9150	9143	1	2	3	5	6
24	·9135	9128	9121	9114	9107	9100	9092	9085	9078	9070	1	2	4	5	6
25	·9063	9056	9048	9041	9033	9026	9018	9011	9003	8996	1	3	4	5	6
26	·8988	8980	8973	8965	8957	8949	8942	8934	8926	8918	1	3	4	5	6
27	·8910	8902	8894	8886	8878	8870	8862	8854	8846	8838	1	3	4	5	7
28	·8829	8821	8813	8805	8796	8788	8780	8771	8763	8755	1	3	4	6	7
29	·8746	8738	8729	8721	8712	8704	8695	8686	8678	8669	1	3	4	6	7
30	·8660	8652	8643	8634	8625	8616	8607	8599	8590	8581	1	3	4	6	7
31	·8572	8563	8554	8545	8536	8526	8517	8508	8499	8490	2	3	5	6	8
32	·8480	8471	8462	8453	8443	8434	8425	8415	8406	8396	2	3	5	6	8
33	·8387	8377	8368	8358	8348	8339	8329	8320	8310	8300	2	3	5	6	8
34	·8290	8281	8271	8261	8251	8241	8231	8221	8211	8202	2	3	5	7	8
35	·8192	8181	8171	8161	8151	8141	8131	8121	8111	8100	2	3	5	7	8
36	·8090	8080	8070	8059	8049	8039	8028	8018	8007	7997	2	3	5	7	9
37	·7986	7976	7965	7955	7944	7934	7923	7912	7902	7891	2	4	5	7	9
38	·7880	7869	7859	7848	7837	7826	7815	7804	7793	7782	2	4	5	7	9
39	·7771	7760	7749	7738	7727	7716	7705	7694	7683	7672	2	4	6	7	9
40	·7660	7649	7638	7627	7615	7604	7593	7581	7570	7559	2	4	6	8	9
41	·7547	7536	7524	7513	7501	7490	7478	7466	7455	7443	2	4	6	8	10
42	·7431	7420	7408	7396	7385	7373	7361	7349	7337	7325	2	4	6	8	10
43	·7314	7302	7290	7278	7266	7254	7242	7230	7218	7206	2	4	6	8	10
44	·7193	7181	7169	7157	7145	7133	7120	7108	7096	7083	2	4	6	8	10

[Numbers in difference columns to be subtracted, not added.]

Degrees	0′ 0°·0	6′ 0°·1	12′ 0°·2	18′ 0°·3	24′ 0°·4	30′ 0°·5	36′ 0°·6	42′ 0°·7	48′ 0°·8	54′ 0°·9	Mean Differences 1	2	3	4	5
45	·7071	7059	7046	7034	7022	7009	6997	6984	6972	6959	2	4	6	8	10
46	·6947	6934	6921	6909	6896	6884	6871	6858	6845	6833	2	4	6	8	11
47	·6820	6807	6794	6782	6769	6756	6743	6730	6717	6704	2	4	6	9	11
48	·6691	6678	6665	6652	6639	6626	6613	6600	6587	6574	2	4	7	9	11
49	·6561	6547	6534	6521	6508	6494	6481	6468	6455	6441	2	4	7	9	11
50	·6428	6414	6401	6388	6374	6361	6347	6334	6320	6307	2	4	7	9	11
51	·6293	6280	6266	6252	6239	6225	6211	6198	6184	6170	2	5	7	9	11
52	·6157	6143	6129	6115	6101	6088	6074	6060	6046	6032	2	5	7	9	12
53	·6018	6004	5990	5976	5962	5948	5934	5920	5906	5892	2	5	7	9	12
54	·5878	5864	5850	5835	5821	5807	5793	5779	5764	5750	2	5	7	9	12
55	·5736	5721	5707	5693	5678	5664	5650	5635	5621	5606	2	5	7	10	12
56	·5592	5577	5563	5548	5534	5519	5505	5490	5476	5461	2	5	7	10	12
57	·5446	5432	5417	5402	5388	5373	5358	5344	5329	5314	2	5	7	10	12
58	·5299	5284	5270	5255	5240	5225	5210	5195	5180	5165	2	5	7	10	12
59	·5150	5135	5120	5105	5090	5075	5060	5045	5030	5015	3	5	8	10	13
60	·5000	4985	4970	4955	4939	4924	4909	4894	4879	4863	3	5	8	10	13
61	·4848	4833	4818	4802	4787	4772	4756	4741	4726	4710	3	5	8	10	13
62	·4695	4679	4664	4648	4633	4617	4602	4586	4571	4555	3	5	8	10	13
63	·4540	4524	4509	4493	4478	4462	4446	4431	4415	4399	3	5	8	10	13
64	·4384	4368	4352	4337	4321	4305	4289	4274	4258	4242	3	5	8	11	13
65	·4226	4210	4195	4179	4163	4147	4131	4115	4099	4083	3	5	8	11	13
66	·4067	4051	4035	4019	4003	3987	3971	3955	3939	3923	3	5	8	11	14
67	·3907	3891	3875	3859	3843	3827	3811	3795	3778	3762	3	5	8	11	14
68	·3746	3730	3714	3697	3681	3665	3649	3633	3616	3600	3	5	8	11	14
69	·3584	3567	3551	3535	3518	3502	3486	3469	3453	3437	3	5	8	11	14
70	·3420	3404	3387	3371	3355	3338	3322	3305	3289	3272	3	5	8	11	14
71	·3256	3239	3223	3206	3190	3173	3156	3140	3123	3107	3	6	8	11	14
72	·3090	3074	3057	3040	3024	3007	2990	2974	2957	2940	3	6	8	11	14
73	·2924	2907	2890	2874	2857	2840	2823	2807	2790	2773	3	6	8	11	14
74	·2756	2740	2723	2706	2689	2672	2656	2639	2622	2605	3	6	8	11	14
75	·2588	2571	2554	2538	2521	2504	2487	2470	2453	2436	3	6	8	11	14
76	·2419	2402	2385	2368	2351	2334	2317	2300	2284	2267	3	6	8	11	14
77	·2250	2233	2215	2198	2181	2164	2147	2130	2113	2096	3	6	9	11	14
78	·2079	2062	2045	2028	2011	1994	1977	1959	1942	1925	3	6	9	11	14
79	·1908	1891	1874	1857	1840	1822	1805	1788	1771	1754		6	9	11	14
80	·1736	1719	1702	1685	1668	1650	1633	1616	1599	1582	3	6	9	12	14
81	·1564	1547	1530	1513	1495	1478	1461	1444	1426	1409	3	6	9	12	14
82	·1392	1374	1357	1340	1323	1305	1288	1271	1253	1236	3	6	9	12	14
83	·1219	1201	1184	1167	1149	1132	1115	1097	1080	1063	3	6	9	12	14
84	·1045	1028	1011	0993	0976	0958	0941	0924	0906	0889	3	6	9	12	14
85	·0872	0854	0837	0819	0802	0785	0767	0750	0732	0715	3	6	9	12	15
86	·0698	0680	0663	0645	0628	0610	0593	0576	0558	0541	3	6	9	12	15
87	·0523	0506	0488	0471	0454	0436	0419	0401	0384	0366	3	6	9	12	[illegible]
88	·0349	0332	0314	0297	0279	0262	0244	0227	0209	0192	3	6	9	12	15
89	·0175	0157	0140	0122	0105	0087	0070	0052	0035	0017	3	6	9	12	15
90	·0000														

Degrees	0′ 0°·0	6′ 0°·1	12′ 0°·2	18′ 0°·3	24′ 0°·4	30′ 0°·5	36′ 0°·6	42′ 0°·7	48′ 0°·8	54′ 0°·9	Mean Differences 1	2	3	4	5
0	·0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	3	6	9	12	15
1	·0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	3	6	9	12	15
2	·0349	0367	0384	0402	0419	0437	0454	0472	0489	0507	3	6	9	12	15
3	·0524	0542	0559	0577	0594	0612	0629	0647	0664	0682	3	6	9	12	15
4	·0699	0717	0734	0752	0769	0787	0805	0822	0840	0857	3	6	9	12	15
5	·0875	0892	0910	0928	0945	0963	0981	0998	1016	1033	3	6	9	12	15
6	·1051	1069	1086	1104	1122	1139	1157	1175	1192	1210	3	6	9	12	15
7	·1228	1246	1263	1281	1299	1317	1334	1352	1370	1388	3	6	9	12	15
8	·1405	1423	1441	1459	1477	1495	1512	1530	1548	1566	3	6	9	12	15
9	·1584	1602	1620	1638	1655	1673	1691	1709	1727	1745	3	6	9	12	15
10	·1763	1781	1799	1817	1835	1853	1871	1890	1908	1926	3	6	9	12	15
11	·1944	1962	1980	1998	2016	2035	2053	2071	2089	2107	3	6	9	12	15
12	·2126	2144	2162	2180	2199	2217	2235	2254	2272	2290	3	6	9	12	15
13	·2309	2327	2345	2364	2382	2401	2419	2438	2456	2475	3	6	9	12	15
14	·2493	2512	2530	2549	2568	2586	2605	2623	2642	2661	3	6	9	12	16
15	·2679	2698	2717	2736	2754	2773	2792	2811	2830	2849	3	6	9	13	16
16	·2867	2886	2905	2924	2943	2962	2981	3000	3019	3038	3	6	9	13	16
17	·3057	3076	3096	3115	3134	3153	3172	3191	3211	3230	3	6	10	13	16
18	·3249	3269	3288	3307	3327	3346	3365	3385	3404	3424	3	6	10	13	16
19	·3443	3463	3482	3502	3522	3541	3561	3581	3600	3620	3	7	10	13	16
20	·3640	3659	3679	3699	3719	3739	3759	3779	3799	3819	3	7	10	13	17
21	·3839	3859	3879	3899	3919	3939	3959	3979	4000	4020	3	7	10	13	17
22	·4040	4061	4081	4101	4122	4142	4163	4183	4204	4224	3	7	10	14	17
23	·4245	4265	4286	4307	4327	4348	4369	4390	4411	4431	3	7	10	14	17
24	·4452	4473	4494	4515	4536	4557	4578	4599	4621	4642	4	7	11	14	18
25	·4663	4684	4706	4727	4748	4770	4791	4813	4834	4856	4	7	11	14	18
26	·4877	4899	4921	4942	4964	4986	5008	5029	5051	5073	4	7	11	15	18
27	·5095	5117	5139	5161	5184	5206	5228	5250	5272	5295	4	7	11	15	18
28	·5317	5340	5362	5384	5407	5430	5452	5475	5498	5520	4	8	11	15	19
29	·5543	5566	5589	5612	5635	5658	5681	5704	5727	5750	4	8	12	15	19
30	·5774	5797	5820	5844	5867	5890	5914	5938	5961	5985	4	8	12	16	20
31	·6009	6032	6056	6080	6104	6128	6152	6176	6200	6224	4	8	12	16	20
32	·6249	6273	6297	6322	6346	6371	6395	6420	6445	6469	4	8	12	16	20
33	·6494	6519	6544	6569	6594	6619	6644	6669	6694	6720	4	8	13	17	21
34	·6745	6771	6796	6822	6847	6873	6899	6924	6950	6976	4	9	13	17	21
35	·7002	7028	7054	7080	7107	7133	7159	7186	7212	7239	4	9	13	18	22
36	·7265	7292	7319	7346	7373	7400	7427	7454	7481	7508	5	9	14	18	23
37	·7536	7563	7590	7618	7646	7673	7701	7729	7757	7785	5	9	14	18	23
38	·7813	7841	7869	7898	7926	7954	7983	8012	8040	8069	5	9	14	19	24
39	·8098	8127	8156	8185	8214	8243	8273	8302	8332	8361	5	10	15	20	24
40	·8391	8421	8451	8481	8511	8541	8571	8601	8632	8662	5	10	15	20	25
41	·8693	8724	8754	8785	8816	8847	8878	8910	8941	8972	5	10	16	21	26
42	·9004	9036	9067	9099	9131	9163	9195	9228	9260	9293	5	11	16	21	27
43	·9325	9358	9391	9424	9457	9490	9523	9556	9590	9623	6	11	17	22	28
44	·9657	9691	9725	9759	9793	9827	9861	9896	9930	9965	6	11	17	23	29

NATURAL TANGENTS

Degrees	0′ 0°·0	6′ 0°·1	12′ 0°·2	18′ 0°·3	24′ 0°·4	30′ 0°·5	36′ 0°·6	42′ 0°·7	48′ 0°·8	54′ 0°·9	Mean Differences 1	2	3	4	5
45	1·0000	0035	0070	0105	0141	0176	0212	0247	0283	0319	6	12	18	24	30
46	1·0355	0392	0428	0464	0501	0538	0575	0612	0649	0686	6	12	18	25	31
47	1·0724	0761	0799	0837	0875	0913	0951	0990	1028	1067	6	13	19	25	32
48	1·1106	1145	1184	1224	1263	1303	1343	1383	1423	1463	7	13	20	27	33
49	1·1504	1544	1585	1626	1667	1708	1750	1792	1833	1875	7	14	21	28	34
50	1·1918	1960	2002	2045	2088	2131	2174	2218	2261	2305	7	14	22	29	36
51	1·2349	2393	2437	2482	2527	2572	2617	2662	2708	2753	8	15	23	30	38
52	1·2799	2846	2892	2938	2985	3032	3079	3127	3175	3222	8	16	24	31	39
53	1·3270	3319	3367	3416	3465	3514	3564	3613	3663	3713	8	16	25	33	41
54	1·3764	3814	3865	3916	3968	4019	4071	4124	4176	4229	9	17	26	34	43
55	1·4281	4335	4388	4442	4496	4550	4605	4659	4715	4770	9	18	27	36	45
56	1·4826	4882	4938	4994	5051	5108	5166	5224	5282	5340	10	19	29	38	48
57	1·5399	5458	5517	5577	5637	5697	5757	5818	5880	5941	10	20	30	40	50
58	1·6003	6066	6128	6191	6255	6319	6383	6447	6512	6577	11	21	32	43	53
59	1·6643	6709	6775	6842	6909	6977	7045	7113	7182	7251	11	23	34	45	56
60	1·7321	7391	7461	7532	7603	7675	7747	7820	7893	7966	12	24	36	48	60
61	1·8040	8115	8190	8265	8341	8418	8495	8572	8650	8728	13	26	38	51	64
62	1·8807	8887	8967	9047	9128	9210	9292	9375	9458	9542	14	27	41	55	68
63	1·9626	9711	9797	9883	9970	2·0057	2·0145	2·0233	2·0323	2·0413	15	29	44	58	73
64	2·0503	0594	0686	0778	0872	0965	1060	1155	1251	1348	16	31	47	63	78
65	2·1445	1543	1642	1742	1842	1943	2045	2148	2251	2355	17	34	51	68	85
66	2·2460	2566	2673	2781	2889	2998	3109	3220	3332	3445	18	37	55	73	92
67	2·3559	3673	3789	3906	4023	4142	4262	4383	4504	4627	20	40	60	79	99
68	2·4751	4876	5002	5129	5257	5386	5517	5649	5782	5916	22	43	65	87	108
69	2·6051	6187	6325	6464	6605	6746	6889	7034	7179	7326	24	47	71	95	119
70	2·7475	7625	7776	7929	8083	8239	8397	8556	8716	8878	26	52	78	104	131
71	2·9042	9208	9375	9544	9714	9887	3·0061	3·0237	3·0415	3·0595	29	58	87	116	145
72	3·0777	0961	1146	1334	1524	1716	1910	2106	2305	2506	32	64	96	129	161
73	3·2709	2914	3122	3332	3544	3759	3977	4197	4420	4646	36	72	108	144	180
74	3·4874	5105	5339	5576	5816	6059	6305	6554	6806	7062	41	81	122	163	204
75	3·7321	7583	7848	8118	8391	8667	8947	9232	9520	9812	46	93	139	186	232
76	4·0108	0408	0713	1022	1335	1653	1976	2303	2635	2972	53	107	160	213	267
77	4·3315	3662	4015	4374	4737	5107	5483	5864	6252	6646	Mean differences cease to be sufficiently accurate.				
78	4·7046	7453	7867	8288	8716	9152	9594	5·0045	5·0504	5·0970					
79	5·1446	1929	2422	2924	3435	3955	4486	5026	5578	6140					
80	5·6713	7297	7894	8502	9124	9758	6·0405	6·1066	6·1742	6·2432					
81	6·3138	3859	4596	5350	6122	6912	7720	8548	9395	7·0264					
82	7·1154	2066	3002	3962	4947	5958	6996	8062	9158	8·0285					
83	8·1443	2636	3863	5126	6427	7769	9152	9·0579	9·2052	9·3572					
84	9·5144	9·677	9·845	10·02	10·20	10·39	10·58	10·78	10·99	11·20					
85	11·43	11·66	11·91	12·16	12·43	12·71	13·00	13·30	13·62	13·95					
86	14·30	14·67	15·06	15·46	15·89	16·35	16·83	17·34	17·89	18·46					
87	19·08	19·74	20·45	21·20	22·02	22·90	23·86	24·90	26·03	27·27					
88	28·64	30·14	31·82	33·69	35·80	38·19	40·92	44·07	47·74	52·08					
89	57·29	63·66	71·62	81·85	95·49	114·6	143·2	191·0	286·5	573·0					
90	∞														

LOGARITHMS OF SINES

Degrees	0′ 0°·0	6′ 0°·1	12′ 0°·2	18′ 0°·3	24′ 0°·4	30′ 0°·5	36′ 0°·6	42′ 0°·7	48′ 0°·8	54′ 0°·9	Mean Differences 1	2	3	4	5
0	∞	$\bar{3}$·2419	$\bar{3}$·5429	7190	8439	9408	$\bar{2}$·0200	$\bar{2}$·0870	$\bar{2}$·1450	$\bar{2}$·1961					
1	$\bar{2}$·2419	2832	3210	3558	3880	4179	4459	4723	4971	5206					
2	$\bar{2}$·5428	5640	5842	6035	6220	6397	6567	6731	6889	7041					
3	$\bar{2}$·7188	7330	7468	7602	7731	7857	7979	8098	8213	8326					
4	$\bar{2}$·8436	8543	8647	8749	8849	8946	9042	9135	9226	9315	16	32	48	64	80
5	$\bar{2}$·9403	9489	9573	9655	9736	9816	9894	9970	$\bar{1}$·0046	$\bar{1}$·0120	13	26	39	52	65
6	$\bar{1}$·0192	0264	0334	0403	0472	0539	0605	0670	0734	0797	11	22	33	44	55
7	$\bar{1}$·0859	0920	0981	1040	1099	1157	1214	1271	1326	1381	10	19	29	38	48
8	$\bar{1}$·1436	1489	1542	1594	1646	1697	1747	1797	1847	1895	8	17	25	34	42
9	$\bar{1}$·1943	1991	2038	2085	2131	2176	2221	2266	2310	2353	8	15	23	30	38
10	$\bar{1}$·2397	2439	2482	2524	2565	2606	2647	2687	2727	2767	7	14	20	27	34
11	$\bar{1}$·2806	2845	2883	2921	2959	2997	3034	3070	3107	3143	6	12	19	25	31
12	$\bar{1}$·3179	3214	3250	3284	3319	3353	3387	3421	3455	3488	6	11	17	23	28
13	$\bar{1}$·3521	3554	3586	3618	3650	3682	3713	3745	3775	3806	5	11	16	21	26
14	$\bar{1}$·3837	3867	3897	3927	3957	3986	4015	4044	4073	4102	5	10	15	20	24
15	$\bar{1}$·4130	4158	4186	4214	4242	4269	4296	4323	4350	4377	5	9	14	18	23
16	$\bar{1}$·4403	4430	4456	4482	4508	4533	4559	4584	4609	4634	4	9	13	17	21
17	$\bar{1}$·4659	4684	4709	4733	4757	4781	4805	4829	4853	4876	4	8	12	16	20
18	$\bar{1}$·4900	4923	4946	4969	4992	5015	5037	5060	5082	5104	4	8	11	15	19
19	$\bar{1}$·5126	5148	5170	5192	5213	5235	5256	5278	5299	5320	4	7	11	14	18
20	$\bar{1}$·5341	5361	5382	5402	5423	5443	5463	5484	5504	5523	3	7	10	14	17
21	$\bar{1}$·5543	5563	5583	5602	5621	5641	5660	5679	5698	5717	3	6	10	13	16
22	$\bar{1}$·5736	5754	5773	5792	5810	5828	5847	5865	5883	5901	3	6	9	12	15
23	$\bar{1}$·5919	5937	5954	5972	5990	6007	6024	6042	6059	6076	3	6	9	12	15
24	$\bar{1}$·6093	6110	6127	6144	6161	6177	6194	6210	6227	6243	3	6	8	11	14
25	$\bar{1}$·6259	6276	6292	6308	6324	6340	6356	6371	6387	6403	3	5	8	11	13
26	$\bar{1}$·6418	6434	6449	6465	6480	6495	6510	6526	6541	6556	3	5	8	10	13
27	$\bar{1}$·6570	6585	6600	6615	6629	6644	6659	6673	6687	6702	2	5	7	10	12
28	$\bar{1}$·6716	6730	6744	6759	6773	6787	6801	6814	6828	6842	2	5	7	9	12
29	$\bar{1}$·6856	6869	6883	6896	6910	6923	6937	6950	6963	6977	2	4	7	9	11
30	$\bar{1}$·6990	7003	7016	7029	7042	7055	7068	7080	7093	7106	2	4	6	9	11
31	$\bar{1}$·7118	7131	7144	7156	7168	7181	7193	7205	7218	7230	2	4	6	8	10
32	$\bar{1}$·7242	7254	7266	7278	7290	7302	7314	7326	7338	7349	2	4	6	8	10
33	$\bar{1}$·7361	7373	7384	7396	7407	7419	7430	7442	7453	7464	2	4	6	8	10
34	$\bar{1}$·7476	7487	7498	7509	7520	7531	7542	7553	7564	7575	2	4	6	7	9
35	$\bar{1}$·7586	7597	7607	7618	7629	7640	7650	7661	7671	7682	2	4	5	7	9
36	$\bar{1}$·7692	7703	7713	7723	7734	7744	7754	7764	7774	7785	2	3	5	7	9
37	$\bar{1}$·7795	7805	7815	7825	7835	7844	7854	7864	7874	7884	2	3	5	7	8
38	$\bar{1}$·7893	7903	7913	7922	7932	7941	7951	7960	7970	7979	2	3	5	6	8
39	$\bar{1}$·7989	7998	8007	8017	8026	8035	8044	8053	8063	8072	2	3	5	6	8
40	$\bar{1}$·8081	8090	8099	8108	8117	8125	8134	8143	8152	8161	1	3	4	6	7
41	$\bar{1}$·8169	8178	8187	8195	8204	8213	8221	8230	8238	8247	1	3	4	6	7
42	$\bar{1}$·8255	8264	8272	8280	8289	8297	8305	8313	8322	8330	1	3	4	6	7
43	$\bar{1}$·8338	8346	8354	8362	8370	8378	8386	8394	8402	8410	1	3	4	5	7
44	$\bar{1}$·8418	8426	8433	8441	8449	8457	8464	8472	8480	8487	1	3	4	5	6

Degrees	0′ 0°·0	6′ 0°·1	12′ 0°·2	18′ 0°·3	24′ 0°·4	30′ 0°·5	36′ 0°·6	42′ 0°·7	48′ 0°·8	54′ 0°·9	Differences 1	2	3	4	5
45	1̄·8495	8502	8510	8517	8525	8532	8540	8547	8555	8562	1	2	4	5	6
46	1̄·8569	8577	8584	8591	8598	8606	8613	8620	8627	8634	1	2	4	5	6
47	1̄·8641	8648	8655	8662	8669	8676	8683	8690	8697	8704	1	2	3	5	6
48	1̄·8711	8718	8724	8731	8738	8745	8751	8758	8765	8771	1	2	3	4	6
49	1̄·8778	8784	8791	8797	8804	8810	8817	8823	8830	8836	1	2	3	4	5
50	1̄·8843	8849	8855	8862	8868	8874	8880	8887	8893	8899	1	2	3	4	5
51	1̄·8905	8911	8917	8923	8929	8935	8941	8947	8953	8959	1	2	3	4	5
52	1̄·8965	8971	8977	8983	8989	8995	9000	9006	9012	9018	1	2	3	4	5
53	1̄·9023	9029	9035	9041	9046	9052	9057	9063	9069	9074	1	2	3	4	5
54	1̄·9080	9085	9091	9096	9101	9107	9112	9118	9123	9128	1	2	3	4	5
55	1̄·9134	9139	9144	9149	9155	9160	9165	9170	9175	9181	1	2	3	3	4
56	1̄·9186	9191	9196	9201	9206	9211	9216	9221	9226	9231	1	2	3	3	4
57	1̄·9236	9241	9246	9251	9255	9260	9265	9270	9275	9279	1	2	2	3	4
58	1̄·9284	9289	9294	9298	9303	9308	9312	9317	9322	9326	1	2	2	3	4
59	1̄·9331	9335	9340	9344	9349	9353	9358	9362	9367	9371	1	1	2	3	4
60	1̄·9375	9380	9384	9388	9393	9397	9401	9406	9410	9414	1	1	2	3	4
61	1̄·9418	9422	9427	9431	9435	9439	9443	9447	9451	9455	1	1	2	3	3
62	1̄·9459	9463	9467	9471	9475	9479	9483	9487	9491	9495	1	1	2	3	3
63	1̄·9499	9503	9506	9510	9514	9518	9522	9525	9529	9533	1	1	2	3	3
64	1̄·9537	9540	9544	9548	9551	9555	9558	9562	9566	9569	1	1	2	2	3
65	1̄·9573	9576	9580	9583	9587	9590	9594	9597	9601	9604	1	1	2	2	3
66	1̄·9607	9611	9614	9617	9621	9624	9627	9631	9634	9637	1	1	2	2	3
67	1̄·9640	9643	9647	9650	9653	9656	9659	9662	9666	9669	1	1	2	2	3
68	1̄·9672	9675	9678	9681	9684	9687	9690	9693	9696	9699	0	1	1	2	2
69	1̄·9702	9704	9707	9710	9713	9716	9719	9722	9724	9727	0	1	1	2	2
70	1̄·9730	9733	9735	9738	9741	9743	9746	9749	9751	9754	0	1	1	2	2
71	1̄·9757	9759	9762	9764	9767	9770	9772	9775	9777	9780	0	1	1	2	2
72	1̄·9782	9785	9787	9789	9792	9794	9797	9799	9801	9804	0	1	1	2	2
73	1̄·9806	9808	9811	9813	9815	9817	9820	9822	9824	9826	0	1	1	2	2
74	1̄·9828	9831	9833	9835	9837	9839	9841	9843	9845	9847	0	1	1	1	2
75	1̄·9849	9851	9853	9855	9857	9859	9861	9863	9865	9867	0	1	1	1	2
76	1̄·9869	9871	9873	9875	9876	9878	9880	9882	9884	9885	0	1	1	1	2
77	1̄·9887	9889	9891	9892	9894	9896	9897	9899	9901	9902	0	1	1	1	1
78	1̄·9904	9906	9907	9909	9910	9912	9913	9915	9916	9918	0	1	1	1	1
79	1̄·9919	9921	9922	9924	9925	9927	9928	9929	9931	9932	0	0	1	1	1
80	1̄·9934	9935	9936	9937	9939	9940	9941	9943	9944	9945	0	0	1	1	1
81	1̄·9946	9947	9949	9950	9951	9952	9953	9954	9955	9956	0	0	1	1	1
82	1̄·9958	9959	9960	9961	9962	9963	9964	9965	9966	9967	0	0	1	1	1
83	1̄·9968	9968	9969	9970	9971	9972	9973	9974	9975	9975	0	0	0	1	1
84	1̄·9976	9977	9978	9978	9979	9980	9981	9981	9982	9983	0	0	0	0	1
85	1̄·9983	9984	9985	9985	9986	9987	9987	9988	9988	9989	0	0	0	0	0
86	1̄·9989	9990	9990	9991	9991	9992	9992	9993	9993	9994	0	0	0	0	0
87	1̄·9994	9994	9995	9995	9996	9996	9996	9997	9997	9997	0	0	0	0	0
88	1̄·9997	9998	9998	9998	9998	9999	9999	9999	9999	9999	0	0	0	0	0
89	1̄·9999	9999	0·0000	0000	0000	0000	0000	0000	0000	0000					
90	0·0000														

LOGARITHMS OF COSINES

[Numbers in difference columns to be subtracted, not added.]

Degrees	0′ 0°·0	6′ 0°·1	12′ 0°·2	18′ 0°·3	24′ 0°·4	30′ 0°·5	36′ 0°·6	42′ 0°·7	48′ 0°·8	54′ 0°·9	Mean Differences 1	2	3	4	5
0	0·0000	0000	0000	0000	0000	0000	0000	0000	0000	$\bar{1}$·9999	0	0	0	0	0
1	$\bar{1}$·9999	9999	9999	9999	9999	9999	9998	9998	9998	9998	0	0	0	0	0
2	$\bar{1}$·9997	9997	9997	9996	9996	9996	9996	9995	9995	9994	0	0	0	0	0
3	$\bar{1}$·9994	9994	9993	9993	9992	9992	9991	9991	9990	9990	0	0	0	0	0
4	$\bar{1}$·9989	9989	9988	9988	9987	9987	9986	9985	9985	9984	0	0	0	0	0
5	$\bar{1}$·9983	9983	9982	9981	9981	9980	9979	9978	9978	9977	0	0	0	0	1
6	$\bar{1}$·9976	9975	9975	9974	9973	9972	9971	9970	9969	9968	0	0	0	1	1
7	$\bar{1}$·9968	9967	9966	9965	9964	9963	9962	9961	9960	9959	0	0	1	1	1
8	$\bar{1}$·9958	9956	9955	9954	9953	9952	9951	9950	9949	9947	0	0	1	1	1
9	$\bar{1}$·9946	9945	9944	9943	9941	9940	9939	9937	9936	9935	0	0	1	1	1
10	$\bar{1}$·9934	9932	9931	9929	9928	9927	9925	9924	9922	9921	0	0	1	1	1
11	$\bar{1}$·9919	9918	9916	9915	9913	9912	9910	9909	9907	9906	0	1	1	1	1
12	$\bar{1}$·9904	9902	9901	9899	9897	9896	9894	9892	9891	9889	0	1	1	1	1
13	$\bar{1}$·9887	9885	9884	9882	9880	9878	9876	9875	9873	9871	0	1	1	1	2
14	$\bar{1}$·9869	9867	9865	9863	9861	9859	9857	9855	9853	9851	0	1	1	1	2
15	$\bar{1}$·9849	9847	9845	9843	9841	9839	9837	9835	9833	9831	0	1	1	1	2
16	$\bar{1}$·9828	9826	9824	9822	9820	9817	9815	9813	9811	9808	0	1	1	2	2
17	$\bar{1}$·9806	9804	9801	9799	9797	9794	9792	9789	9787	9785	0	1	1	2	2
18	$\bar{1}$·9782	9780	9777	9775	9772	9770	9767	9764	9762	9759	0	1	1	2	2
19	$\bar{1}$·9757	9754	9751	9749	9746	9743	9741	9738	9735	9733	0	1	1	2	2
20	$\bar{1}$·9730	9727	9724	9722	9719	9716	9713	9710	9707	9704	0	1	1	2	2
21	$\bar{1}$·9702	9699	9696	9693	9690	9687	9684	9681	9678	9675	0	1	1	2	2
22	$\bar{1}$·9672	9669	9666	9662	9659	9656	9653	9650	9647	9643	1	1	2	2	3
23	$\bar{1}$·9640	9637	9634	9631	9627	9624	9621	9617	9614	9611	1	1	2	2	3
24	$\bar{1}$·9607	9604	9601	9597	9594	9590	9587	9583	9580	9576	1	1	2	2	3
25	$\bar{1}$·9573	9569	9566	9562	9558	9555	9551	9548	9544	9540	1	1	2	2	3
26	$\bar{1}$·9537	9533	9529	9525	9522	9518	9514	9510	9506	9503	1	1	2	3	3
27	$\bar{1}$·9499	9495	9491	9487	9483	9479	9475	9471	9467	9463	1	1	2	3	3
28	$\bar{1}$·9459	9455	9451	9447	9443	9439	9435	9431	9427	9422	1	1	2	3	3
29	$\bar{1}$·9418	9414	9410	9406	9401	9397	9393	9388	9384	9380	1	1	2	3	4
30	$\bar{1}$·9375	9371	9367	9362	9358	9353	9349	9344	9340	9335	1	1	2	3	4
31	$\bar{1}$·9331	9326	9322	9317	9312	9308	9303	9298	9294	9289	1	2	2	3	4
32	$\bar{1}$·9284	9279	9275	9270	9265	9260	9255	9251	9246	9241	1	2	2	3	4
33	$\bar{1}$·9236	9231	9226	9221	9216	9211	9206	9201	9196	9191	1	2	3	3	4
34	$\bar{1}$·9186	9181	9175	9170	9165	9160	9155	9149	9144	9139	1	2	3	3	4
35	$\bar{1}$·9134	9128	9123	9118	9112	9107	9101	9096	9091	9085	1	2	3	4	5
36	$\bar{1}$·9080	9074	9069	9063	9057	9052	9046	9041	9035	9029	1	2	3	4	5
37	$\bar{1}$·9023	9018	9012	9006	9000	8995	8989	8983	8977	8971	1	2	3	4	5
38	$\bar{1}$·8965	8959	8953	8947	8941	8935	8929	8923	8917	8911	1	2	3	4	5
39	$\bar{1}$·8905	8899	8893	8887	8880	8874	8868	8862	8855	8849	1	2	3	4	5
40	$\bar{1}$·8843	8836	8830	8823	8817	8810	8804	8797	8791	8784	1	2	3	4	5
41	$\bar{1}$·8778	8771	8765	8758	8751	8745	8738	8731	8724	8718	1	2	3	5	6
42	$\bar{1}$·8711	8704	8697	8690	8683	8676	8669	8662	8655	8648	1	2	3	5	6
43	$\bar{1}$·8641	8634	8627	8620	8613	8606	8598	8591	8584	8577	1	2	4	5	6
44	$\bar{1}$·8569	8562	8555	8547	8540	8532	8525	8517	8510	8502	1	2	4	5	6

[Numbers in difference columns to be subtracted, not added.]

Degrees	0′ 0°·0	6′ 0°·1	12′ 0°·2	18′ 0°·3	24′ 0°·4	30′ 0°·5	36′ 0°·6	42′ 0°·7	48′ 0°·8	54′ 0°·9	Mean Differences 1	2	3	4	5
45	$\bar{1}$·8495	8487	8480	8472	8464	8457	8449	8441	8433	8426	1	3	4	5	6
46	$\bar{1}$·8418	8410	8403	8394	8386	8378	8370	8362	8354	8346	1	3	4	5	7
47	$\bar{1}$·8338	8330	8322	8313	8305	8297	8289	8280	8272	8264	1	3	4	6	7
48	$\bar{1}$·8255	8247	8238	8230	8221	8213	8204	8195	8187	8178	1	3	4	6	7
49	$\bar{1}$·8169	8161	8152	8143	8134	8125	8117	8108	8099	8090	1	3	4	6	7
50	$\bar{1}$·8081	8072	8063	8053	8044	8035	8026	8017	8007	7998	2	3	5	6	8
51	$\bar{1}$·7989	7979	7970	7960	7951	7941	7932	7922	7913	7903	2	3	5	6	8
52	$\bar{1}$·7893	7884	7874	7864	7854	7844	7835	7825	7815	7805	2	3	5	7	8
53	$\bar{1}$·7795	7785	7774	7764	7754	7744	7734	7723	7713	7703	2	3	5	7	9
54	$\bar{1}$·7692	7682	7671	7661	7650	7640	7629	7618	7607	7597	2	4	5	7	9
55	$\bar{1}$·7586	7575	7564	7553	7542	7531	7520	7509	7498	7487	2	4	6	7	9
56	$\bar{1}$·7476	7464	7453	7442	7430	7419	7407	7396	7384	7373	2	4	6	8	10
57	$\bar{1}$·7361	7349	7338	7326	7314	7302	7290	7278	7266	7254	2	4	6	8	10
58	$\bar{1}$·7242	7230	7218	7205	7193	7181	7168	7156	7144	7131	2	4	6	8	10
59	$\bar{1}$·7118	7106	7093	7080	7068	7055	7042	7029	7016	7003	2	4	6	9	11
60	$\bar{1}$·6990	6977	6963	6950	6937	6923	6910	6896	6883	6869	2	4	7	9	11
61	$\bar{1}$·6856	6842	6828	6814	6801	6787	6773	6759	6744	6730	2	5	7	9	12
62	$\bar{1}$·6716	6702	6687	6673	6659	6644	6629	6615	6600	6585	2	5	7	10	12
63	$\bar{1}$·6570	6556	6541	6526	6510	6495	6480	6465	6449	6434	3	5	8	10	13
64	$\bar{1}$·6418	6403	6387	6371	6356	6340	6324	6308	6292	6276	3	5	8	11	13
65	$\bar{1}$·6259	6243	6227	6210	6194	6177	6161	6144	6127	6110	3	6	8	11	14
66	$\bar{1}$·6093	6076	6059	6042	6024	6007	5990	5972	5954	5937	3	6	9	12	15
67	$\bar{1}$·5919	5901	5883	5865	5847	5828	5810	5792	5773	5754	3	6	9	12	15
68	$\bar{1}$·5736	5717	5698	5679	5660	5641	5621	5602	5583	5563	3	6	10	13	16
69	$\bar{1}$·5543	5523	5504	5484	5463	5443	5423	5402	5382	5361	3	7	10	14	17
70	$\bar{1}$·5341	5320	5299	5278	5256	5235	5213	5192	5170	5148	4	7	11	14	18
71	$\bar{1}$·5126	5104	5082	5060	5037	5015	4992	4969	4946	4923	4	8	11	15	19
72	$\bar{1}$·4900	4876	4853	4829	4805	4781	4757	4733	4709	4684	4	8	12	16	20
73	$\bar{1}$·4659	4634	4609	4584	4559	4533	4508	4482	4456	4430	4	9	13	17	21
74	$\bar{1}$·4403	4377	4350	4323	4296	4269	4242	4214	4186	4158	5	9	14	18	23
75	$\bar{1}$·4130	4102	4073	4044	4015	3986	3957	3927	3897	3867	5	10	15	20	24
76	$\bar{1}$·3837	3806	3775	3745	3713	3682	3650	3618	3586	3554	5	11	16	21	26
77	$\bar{1}$·3521	3488	3455	3421	3387	3353	3319	3284	3250	3214	6	11	17	23	28
78	$\bar{1}$·3179	3143	3107	3070	3034	2997	2959	2921	2883	2845	6	12	19	25	31
79	$\bar{1}$·2806	2767	2727	2687	2647	2606	2565	2524	2482	2439	7	14	20	27	34
80	$\bar{1}$·2397	2353	2310	2266	2221	2176	2131	2085	2038	1991	8	15	23	30	38
81	$\bar{1}$·1943	1895	1847	1797	1747	1697	1646	1594	1542	1489	8	17	25	34	42
82	$\bar{1}$·1436	1381	1326	1271	1214	1157	1099	1040	0981	0920	10	19	29	38	48
83	$\bar{1}$·0859	0797	0734	0670	0605	0539	0472	0403	0334	0264	11	22	33	44	55
84	$\bar{1}$·0192	0120	0046	$\bar{2}$·9970	$\bar{2}$·9894	$\bar{2}$·9816	$\bar{2}$·9736	$\bar{2}$·9655	$\bar{2}$·9573	$\bar{2}$·9489	13	26	39	52	65
85	$\bar{2}$·9403	9315	9226	9135	9042	8946	8849	8749	8647	8543	16	32	48	64	80
86	$\bar{2}$·8436	8326	8213	8098	7979	7857	7731	7602	7468	7330					
87	$\bar{2}$·7188	7041	6889	6731	6567	6397	6220	6035	5842	5640					
88	$\bar{2}$·5428	5206	4971	4723	4459	4175	3880	3558	3210	2832					
89	$\bar{2}$·2419	1961	1450	0870	0200	$\bar{3}$·9408	$\bar{3}$·8439	$\bar{3}$·7190	$\bar{3}$·5429	$\bar{3}$·2419					
90	$-\infty$														

Degrees	0′ 0°·0	6′ 0°·1	12′ 0°·2	18′ 0°·3	24′ 0°·4	30′ 0°·5	36′ 0°·6	42′ 0°·7	48′ 0°·8	54′ 0°·9	Mean Differences 1	2	3	4	5
0	−∞	$\bar{3}$·2419	$\bar{3}$·5429	$\bar{3}$·7190	$\bar{3}$·8439	$\bar{3}$·9409	$\bar{2}$·0200	$\bar{2}$·0870	$\bar{2}$·1450	$\bar{2}$·1962					
1	$\bar{2}$·2419	2833	3211	3559	3881	4181	4461	4725	4973	5208					
2	$\bar{2}$·5431	5643	5845	6038	6223	6401	6571	6736	6894	7046					
3	$\bar{2}$·7194	7337	7475	7609	7739	7865	7988	8107	8223	8336					
4	$\bar{2}$·8446	8554	8659	8762	8862	8960	9056	9150	9241	9331	16	32	48	64	8
5	$\bar{2}$·9420	9506	9591	9674	9756	9836	9915	9992	$\bar{1}$·0068	$\bar{1}$·0143	13	26	40	53	6
6	$\bar{1}$·0216	0289	0360	0430	0499	0567	0633	0699	0764	0828	11	22	34	45	5
7	$\bar{1}$·0891	0954	1015	1076	1135	1194	1252	1310	1367	1423	10	20	29	39	4
8	$\bar{1}$·1478	1533	1587	1640	1693	1745	1797	1848	1898	1948	9	17	26	35	4
9	$\bar{1}$·1997	2046	2094	2142	2189	2236	2282	2328	2374	2419	8	16	23	31	3
10	$\bar{1}$·2463	2507	2551	2594	2637	2680	2722	2764	2805	2846	7	14	21	28	35
11	$\bar{1}$·2887	2927	2967	3006	3046	3085	3123	3162	3200	3237	6	13	19	26	3
12	$\bar{1}$·3275	3312	3349	3385	3422	3458	3493	3529	3564	3599	6	12	18	24	3
13	$\bar{1}$·3634	3668	3702	3736	3770	3804	3837	3870	3903	3935	6	11	17	22	28
14	$\bar{1}$·3968	4000	4032	4064	4095	4127	4158	4189	4220	4250	5	10	16	21	26
15	$\bar{1}$·4281	4311	4341	4371	4400	4430	4459	4488	4517	4546	5	10	15	20	25
16	$\bar{1}$·4575	4603	4632	4660	4688	4716	4744	4771	4799	4826	5	9	14	19	23
17	$\bar{1}$·4853	4880	4907	4934	4961	4987	5014	5040	5066	5092	4	9	13	18	22
18	$\bar{1}$·5118	5143	5169	5195	5220	5245	5270	5295	5320	5345	4	8	13	17	21
19	$\bar{1}$·5370	5394	5419	5443	5467	5491	5516	5539	5563	5587	4	8	12	16	20
20	$\bar{1}$·5611	5634	5658	5681	5704	5727	5750	5773	5796	5819	4	8	12	15	19
21	$\bar{1}$·5842	5864	5887	5909	5932	5954	5976	5998	6020	6042	4	7	11	15	19
22	$\bar{1}$·6064	6086	6108	6129	6151	6172	6194	6215	6236	6257	4	7	11	14	18
23	$\bar{1}$·6279	6300	6321	6341	6362	6383	6404	6424	6445	6465	3	7	10	14	17
24	$\bar{1}$·6486	6506	6527	6547	6567	6587	6607	6627	6647	6667	3	7	10	13	17
25	$\bar{1}$·6687	6706	6726	6746	6765	6785	6804	6824	6843	6863	3	7	10	13	16
26	$\bar{1}$·6882	6901	6920	6939	6958	6977	6996	7015	7034	7053	3	6	9	13	16
27	$\bar{1}$·7072	7090	7109	7128	7146	7165	7183	7202	7220	7238	3	6	9	12	15
28	$\bar{1}$·7257	7275	7293	7311	7330	7348	7366	7384	7402	7420	3	6	9	12	15
29	$\bar{1}$·7438	7455	7473	7491	7509	7526	7544	7562	7579	7597	3	6	9	12	15
30	$\bar{1}$·7614	7632	7649	7667	7684	7701	7719	7736	7753	77[illegible]	3	6	9	12	14
31	$\bar{1}$·7788	7805	7822	7839	7856	7873	7890	7907	7924	7941	3	6	9	11	14
32	$\bar{1}$·7958	7975	7992	8008	8025	8042	8059	8075	8092	8109	3	6	8	11	14
33	$\bar{1}$·8125	8142	8158	8175	8191	8208	8224	8241	8257	8274	3	5	8	11	14
34	$\bar{1}$·8290	8306	8323	8339	8355	8371	8388	8404	8420	8436	3	5	8	11	14
35	$\bar{1}$·8452	8468	8484	8501	8517	8533	8549	8565	8581	8597	3	5	8	11	13
36	$\bar{1}$·8613	8629	8644	8660	8676	8692	8708	8724	8740	8755	3	5	8	11	13
37	$\bar{1}$·8771	8787	8803	8818	8834	8850	8865	8881	8897	8912	3	5	8	10	13
38	$\bar{1}$·8928	8944	8959	8975	8990	9006	9022	9037	9053	9068	3	5	8	10	[illegible]
39	$\bar{1}$·9084	9099	9115	9130	9146	9161	9176	9192	9207	9223	3	5	8	10	
40	$\bar{1}$·9238	9254	9269	9284	9300	9315	9330	9346	9361	9376	3	5	8	10	
41	$\bar{1}$·9392	9407	9422	9438	9453	9468	9483	9499	9514	9529	3	5	8	10	13
42	$\bar{1}$·9544	9560	9575	9590	9605	9621	9636	9651	9666	9681	3	5	8	10	13
43	$\bar{1}$·9697	9712	9727	9742	9757	9772	9788	9803	9818	9833	3	5	8	10	13
44	$\bar{1}$·9848	9864	9879	9894	9909	9924	9939	9955	9970	9985	3	5	8	10	13